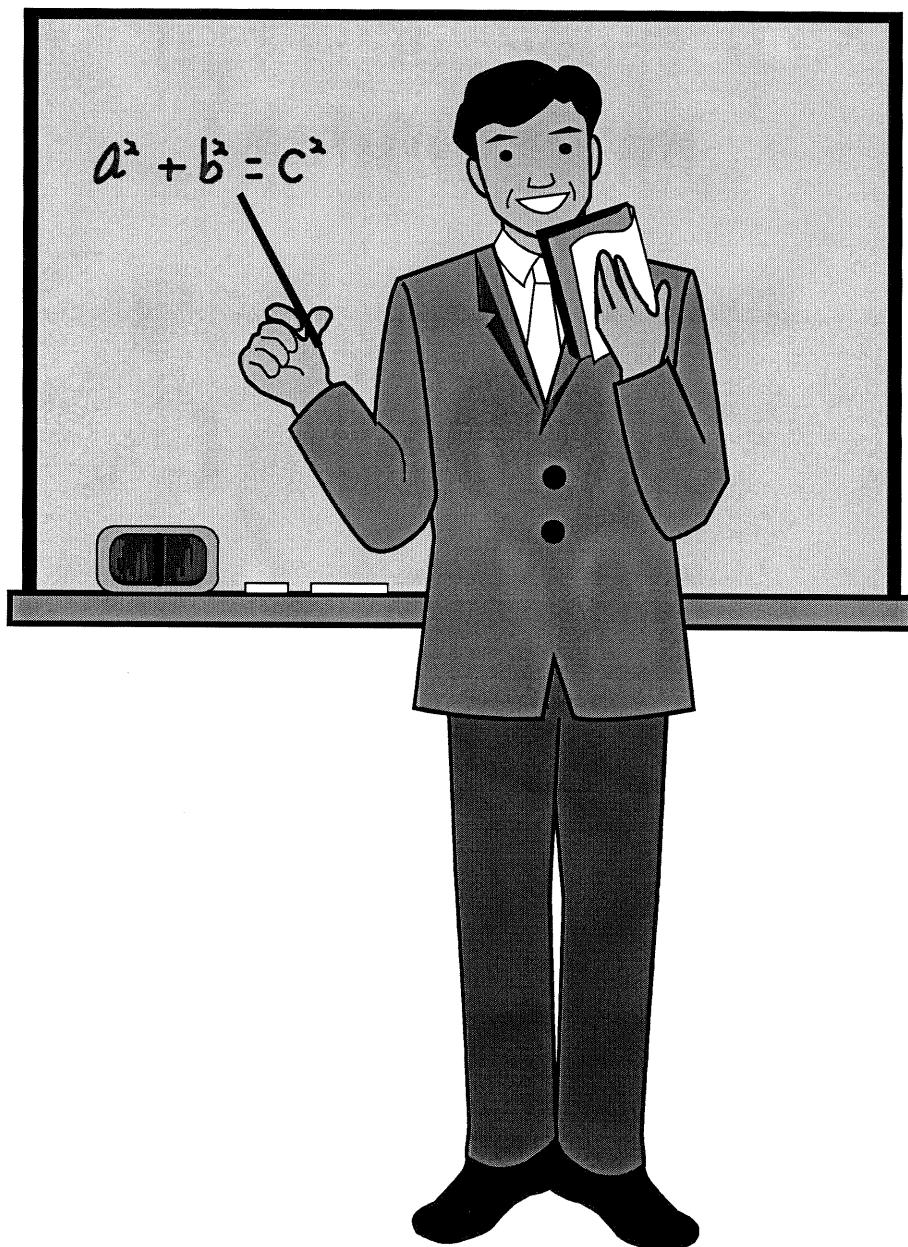


Part 1A NST Mathematics B Course

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Michaelmas Term 2005

Lecture Notes



MATHEMATICAL

METHODS I -B

1. INTRODUCTION

- WELCOME!
- MOSTLY SERVICE COURSE FOR NST
- BUT
 - REALLY IMPORTANT PHYSICS (GRAVITATION HEP..) CHEMISTRY (QUANTUM MECH) BIOLOGY (GENOME ...)
 - SUBJECT IN ITS OWN RIGHT

Abstract ideas in maths are beautiful & often turn out to be essential in science
 - WAVE FUN IN QUANTUM MECHANICS
IS A COMPLEX NUMBER.

What's it all about?

A personal view from Sir Michael Atiyah.

Mathematics is one of the great intellectual achievements of humanity. With a history going back thousands of years it transcends cultural and linguistic barriers and has a permanent value that is impervious to change. It is both an art and a science, having aesthetic and practical qualities. Its utility is not in doubt, as explained in the subsequent pages; its applications spread right across the physical, biological and social sciences. Modern civilisation rests

firmly on a mathematical foundation. On the other hand the driving force that has created this great edifice is human curiosity in its purest form. Once set up, mathematics generates its own internal problems, and it is the continuous attempt to tackle these problems that makes mathematics what it is.

The key principles of mathematics can be described as 'analogy' or 'abstraction'. Structural similarities or analogies between diverse areas can be analysed by concentrating on the essential common features and ignoring the detail – this is the process we call abstraction. For example, wave motion is a term borrowed from the sea, but common to the propagation of light, radio, sound and susceptible

A or B version?

- NOT FINAL TODAY
 $B \rightarrow A$ at any stage
 $A \rightarrow B$ at break points
- MORE EXAM CHOICE IF DO B
- SOME EXTRA MATERIAL IN E
- LIKE MATHS AND/OR DONE MORE THAN SINGLE A LEVEL
- FEEL UP TO CHALLENGE
- PLANNING TO DO PHYSICS AND/OR MATHS NEXT YEAR

BOOKS & SYLLABUS

- SEE HANDOUT

TAKE NOTES ?

You should learn this skill but I will hand out copies of O'hare every ~ 2 weeks

1.2-6

1.2

Mathematical Methods I 24 lectures, Michaelmas term

Vector sum and vector equation of a line. Scalar product, unit vectors, vector equation of a plane. Vector product, vector and scalar triple products. Orthogonal bases. Cartesian components. Spherical and cylindrical polar coordinates.

[5] →

Revision of single variable calculus. Elementary curve sketching. Idea of continuity and differentiability of functions. Orders of magnitude and approximate behaviour for large and small x . The O notation. Leibnitz's formula. The integral as a sum, differentiation of an integral with respect to its limits or a parameter. The approximation of a sum by an integral. Stirling's approximation as an example. Schwarz's inequality.

[5]

Double and triple integrals in Cartesian, spherical and cylindrical coordinates. Examples to include evaluation of $\int \exp(-x^2) dx$

[3]

Power series. Statement of Taylor's theorem. Examples to include the binomial expansion, exponential and trigonometric functions, and logarithm. Newton-Raphson method.

[2]

Complex numbers and complex plane, vector diagrams. Exponential function of a complex variable. $\exp(it)$, complex representations of \cos and \sin . Hyperbolic functions.

[2]

Ordinary differential equations. First order equations: separable equations; linear equations, integrating factors. Second-order linear equations with constant coefficients; $\exp(\lambda x)$ as trial solution, including degenerate case. Superposition. Particular integrals and complementary functions. Constants of integration and number of necessary boundary/initial conditions. Particular integrals by trial solutions.

[5]

Differentiation of functions of several variables. Differentials, chain rule. Unconditional stationary values, maxima and minima.

[2]

BIBLIOGRAPHY

There are very many books which cover the sort of mathematics required by Natural Scientists. The following should be helpful as general reference; further advice will be given by Lecturers.

M L Boas *Mathematical Methods in the Physical Sciences, 2nd edition.* Wiley, 1983 (£31.95 paperback).

A Jeffrey *Mathematics for Engineers and Scientists, 5th edition.* Chapman & Hall, 1996 (£27.25 paperback)

E Kreyszig *Advanced Engineering Mathematics, 8th edition.* Wiley, 1999 (£30.95 paperback).

K F Riley, M P Hobson & S J Bence *Mathematical Methods for Physics and Engineering.* Cambridge University Press, 1998 £21.95 paperback.

I S Sokolnikoff & R M Redheffer *Mathematics of Physics and Modern Engineering, 2nd edition.* McGraw Hill, 1967 £86.99 hardback

G Stephenson *Mathematical Methods for Science Students, 2nd edition.* Longman, 1973 (£27.99 paperback).

G Stephenson *Worked Examples in Mathematics for Scientists and Engineers.* Longman, 1985 (out of print)

K A Stroud & D Booth *Engineering Mathematics, 5th edition.* Palgrave 2001 (£24.99 paperback with CD-ROM)

K A Stroud *Further Engineering Mathematics, 3rd edition.* Palgrave 1996 (£24.99 paperback)

G Thomas & R Finney *Calculus and Analytic Geometry, 9th edition.* Addison-Wesley, 1995 (£62.00 hardback, £33.99 paperback)

1.1 Required Background

- Details of A level courses
NOT critical
- Being reasonably comfortable with algebra IS important.
- Algebraic manipulation is like playing a musical instrument
- You need practice.

Do the examples
Use your supervisors
Use STRAULD
- Spelling out detail in the algebra is main diff between A & B courses. This is why less topics are covered in

R

1.4

- Examples of Algebra I. will tend to assume you find "easy"

$$x^{-1/2} = \frac{1}{\sqrt{x}}$$

$$x^0 = 1$$

$$a^2 - b^2 = (a+b)(a-b)$$

$$\sin(\alpha+\beta) = \sin\alpha \cos\beta + \cos\alpha \sin\beta$$

[- greek letters -]

$$\frac{d}{dx} x^n = nx^{n-1} \quad \text{all } n \quad (\forall n)$$

$$\int x^n = \frac{1}{n+1} x^{n+1} \quad n \neq ?$$

$$ax^2 + bx + c = 0 \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

[- formula book?]

The Greek Alphabet for Maths and Science

Lower case	Upper case	Name	English Equivalent	Lower case	Upper case	Name	English Equivalent
α	A	alpha	A	v	N	nu	N
β	B	beta	B	o	O	omicron	O
χ	X	chi	C	π	II	pi	P
δ	Δ	delta	D	θ	Θ	theta	T
ϵ	E	epsilon	E	ρ	ρ'	rho	R
ϕ	Φ	phi	F	σ	Σ	sigma	S
γ	Γ	gamma	G	τ	T	tau	T
η	H	eta	H	v	Y	upsilon	U
ι	I	iota	I	-	-	-	V
-	-	-	J	ω	Ω	omega	W
κ	K	kappa	K	ξ	Ξ	xi	X
λ	Λ	lambda	L	ψ	Ψ	psi	Y
μ	M	mu	M	ζ	Z	zeta	Z

N.B. this table was produced using Microsoft word and the symbol font for the Greek letters. To typeset complicated mathematics, I recommend using the LaTeX program, however you do not need to find out about this yet.

Many web sites give more detail on the pronunciation of both ancient and modern Greek.

e.g: <http://www.ibiblio.org/koine/greek/lessons/alphabet.html>

Please note scientists do not pretend to use the "correct" pronunciations.

1.2 Proof of Quadratic eqn formula

$$\begin{aligned}
 ax^2 + bx + c &= 0 \\
 \therefore a(x^2 + \frac{b}{a}x) + c &= 0 \\
 \therefore a(x + \frac{b}{2a})^2 - \frac{b^2}{4a} + c &= 0 \\
 \therefore (x + \frac{b}{2a})^2 &= \frac{\frac{b^2}{4a} - c}{a} \\
 (x + \frac{b}{2a})^2 &= \frac{b^2 - 4ac}{4a^2} \\
 x = -\frac{b}{2a} \pm \sqrt{\frac{b^2 - 4ac}{4a^2}} &\quad \underline{\text{QED}}
 \end{aligned}$$

Now see you can do this yourself. If you find it really difficult think about changing to A or spend some quality time with STROUD.

Learning to manage your time will be hard & is REALLY important. 3 hours/week on maths.

2. SOME FORMAL MATHEMATICS

Chapter 2 is NON-Examinable.

2.1 Algebra of Numbers

a, b, c etc represent numbers then:

$$\begin{aligned}
 ab &= ba \\
 a+b &= b+a
 \end{aligned}$$

Addition & multiplication are COMMUTATIVE

$$(ab)c = a(bc)$$

$$(a+b)+c = a+(b+c)$$

ASSOCIATIVE

$$a(b+c) = ab + ac$$

DISTRIBUTIVE

1.3 Other Remarks

- Talk from 9.01 to 9.50 (9.01 not 9.00 is a concession & = loss of 24 mins = $\frac{1}{2}$ lecture).
- Class too big for much dialogue but tell me if I get something wrong or go too fast.
- Remember supervisors are there to help
- Read BOOKS
- 4 Examples classes, chance to review material with me.
- I am happy to get e-mails & will try to answer, (maybe in lectures).

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`Richard.Ansorge@phy.cam.ac.uk`

2.1

2.2 - and ÷

- & ÷ follow same rules as + & × but just a little care is required

$$a - b = -b + a$$

[since $a - b$ is defined as

$$a + (-b)$$

$$= (-b) + a = -b + a]$$

$$\frac{a}{b} = a \times \frac{1}{b} = \frac{1}{b} \times a$$

Of course

$$a - b \neq b - a \quad \text{in general}$$

$$a \div b \neq b \div a$$

order
of
operation
does not
matter

Notice the last one mixes up + & ×

- Actually best to think of $- \& \div$ as **INVERSE OPERATIONS** to $+$ & \times

- i.e.
- adding $(-b)$ undoes the effect of adding b .
 - dividing by b means multiplying by $\frac{1}{b}$ which undoes the effect of multiplying by b .

$$(a+b) - b = a \quad \forall a, b$$

$$(a \cdot b) \div b = a \quad \forall a, b \neq 0$$

$$b - b = 0, \quad \frac{b}{b} = 1 \quad b \neq 0 \quad \left[\begin{array}{l} 0 \\ \text{not} \\ \text{defined} \end{array} \right]$$

0 is null operation w.r.t. $+$, $-$

1 is null operation w.r.t \times , \div

we write $\div b \equiv \frac{1}{b} \equiv [b^{-1}]$

The notation b^{-1} is used quite generally for inverses hence

$$y = \sin \alpha \Leftrightarrow \alpha = \sin^{-1} y$$

2.3 Absolute Proof

- In science we discover things by observation and experiment.
The law of Gravity is "true" because the motion of planets agrees well [Newton \rightarrow Einstein $\rightarrow ?$].
- In maths we prove theorems which are absolutely true. [If no mistake in proof - Fermat's last thm]
- Fascinated mathematicians from Greeks on.
- Not magic, proof comes from applying rules of Logic to Axioms which are assumed to be true.
e.g. $A \rightarrow B$ & $B \rightarrow C$ then $A \rightarrow C$ (log)
parallel lines don't ever meet (axiom)

Beware

$\frac{1}{\sin \alpha}$ is never written as $\sin^{-1} \alpha$

I suppose you could write $\sin^{-1} \alpha$ but that is just not done; you may see $(\sin \alpha)^{-1}$. You may also see $\operatorname{cosec}(\alpha) \equiv \frac{1}{\sin \alpha}$
But I find this too complicated

$$\frac{1}{\cos \alpha} = \sec \alpha \quad \frac{1}{\tan \alpha} = \cot \alpha$$

- One of the themes of this course will be exploring how these rules of algebra carry over to other objects, such as vectors & matrices.
- Mathematicians like to generalize

2.4 Example Pythagoras' Theorem

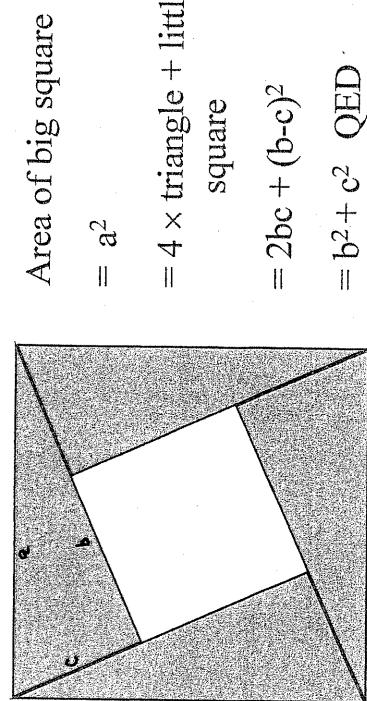
Underlying axioms are the basis of Euclidean Geometry

- Rules for //el lines
 - Similar triangles
- Dont have time to go into all the details.
- Modern proof.
 - Classical proof - see web sites [<http://java.sun.com/applets/archive/beta/Pythagoras>]
- The axioms of Euclidean geometry are consistent. Thus Pythagoras IS TRUE as maths theorem.

The fact that Pythagoras works in the real world is experimental physics!

- i.e. The axioms of Euclidean geometry are (more or less) applicable to our universe.

Simple algebraic proof of Pythagoras

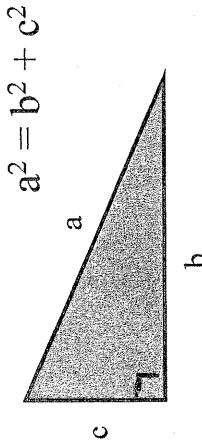


$$\begin{aligned}
 & \text{Area of big square} \\
 & = a^2 \\
 & = 4 \times \text{triangle} + \text{little-square} \\
 & = 2bc + (b-c)^2 \\
 & = b^2 + c^2 \quad \text{QED}
 \end{aligned}$$

Pythagoras Theorem



~560-480 BC

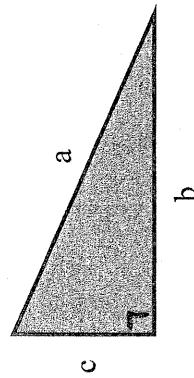


$$a^2 = b^2 + c^2$$

Or: "The square on the hypotenuse is equal to the sum of the squares on the other two sides."

(Statement first found on Babylonian tablet ~1900-1600 BC.)

Pythagoras Theorem



$$c^2 = a^2 + b^2$$

Recall Area of triangle = $\frac{1}{2}$ base \times height

$$= \frac{1}{2} bc$$

2.5 This Course

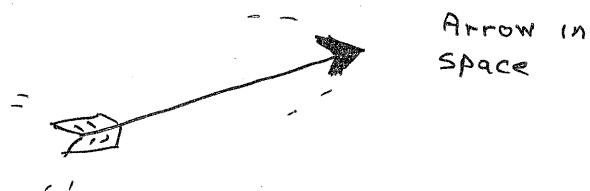
In this course I do not have time to prove everything in all the detail that a mathematician might like.

I will try to at least indicate how statements might be proved.

I will prove some things.

The exams don't ask about proving results - they ask you to do examples using the methods discussed.

3. VECTORS



Think of vectors as objects which have both magnitude & direction.

You have certainly seen many examples - e.g. velocity.

Vectors are actually a new class of mathematical objects we can do algebra with.

It is possible to give a purely mathematical treatment (AXIOMS + THEOREMS) but we will take a more intuitive (physical) approach & emphasise applications.

It is a remarkable fact that nature contains many examples of objects that obey the abstract mathematical rules for vectors i.e. they are vectors.

[Aside:

I think in practice vectors were first discovered by Physicists (Newton). Later mathematicians came along and codified the rules for manipulating vectors and generalized them.

Hence we now have n-dim vector $n=4, 11, \infty \dots$

Example of interaction between maths & physics]

3D World

We live in a 3D world. Most of us find it hard to think in 3 (or more!) dimensions.

Vectors are a big help with 3D problems. Used properly you can often ~~not~~ do a 3D problem without having to think in 3D!

Vectors are essential to go beyond 3D, e.g. 4D space-time.

3.1 Components - later

You may have already seen vectors in your A Level courses.

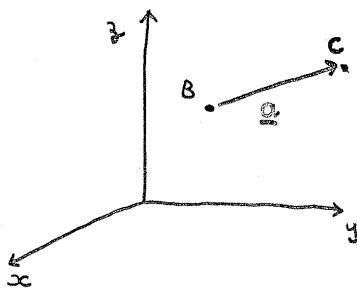
If so you may also have learnt that vectors can be represented by a set of 3 numbers. The components w.r.t. the x, y, z axes.

This is true and useful for calculations and we will come on ~~to~~ to components later.

But a vector is NOT JUST a set of 3 numbers. Please learn to think of a vector as a single object with its own set of rules for manipulation (i.e. addition etc.).

In fact vectors will turn up in all sorts of places and not always with 3 components.

3.2 Displacement vectors



(axes shown
to "frame the
picture")

Notice \underline{q} depends on the relative positions of B & C not on their absolute positions. Many pairs of points might have the same displacement vector \underline{q} (just move B & C ~~the same~~ amount - both distance & dirn).

B & C are 2 points in space

The separation between the points has magnitude & direction

The displacement vector \underline{q} from B to C has magnitude & direction as drawn.

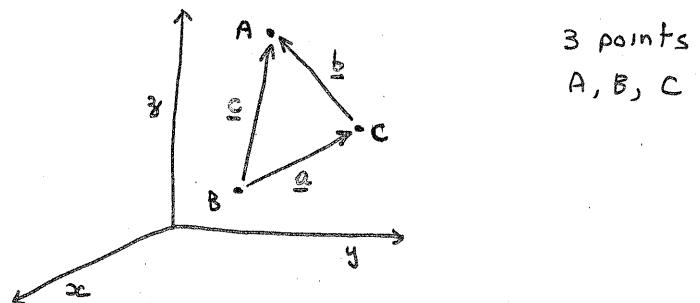
Notation

\underline{q} vector (bold in printed books)

$|q|$ the length of \underline{q} "mod \underline{q} "

\hat{q} a unit length vector // \underline{q} to \underline{q}
"a hat"

3.3 ADDITION OF DISPLACEMENTS



3 points
A, B, C

$$\begin{aligned} \underline{a} &\text{ is displacement } B \rightarrow C \equiv \overrightarrow{BC} \\ \underline{b} &\text{ " " " } \quad C \rightarrow A \equiv \overrightarrow{CA} \\ \underline{c} &\text{ " " " } \quad B \rightarrow A \equiv \overrightarrow{BA} \end{aligned}$$

But to go from B to A we can either:

go directly: $\overrightarrow{BA} = \underline{c}$

or go to C ($\overrightarrow{BC} = \underline{q}$) then go to A ($\overrightarrow{CA} = \underline{b}$)

hence

$$\underline{c} = \underline{q} + \underline{b}$$

This is in fact an operational definition for the addition of 2 vectors,
 $\underline{q} + \underline{b}$ means [first do \underline{q} , then do \underline{b}]

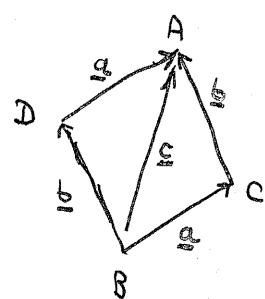
what about $\underline{b} + \underline{q}$? It will turn out that this is also \underline{c} , ie

$$\underline{c} = \underline{q} + \underline{b} = \underline{b} + \underline{q}$$

parallelogram rule for vector addition:

Draw A, B, C in plane of paper & add 4th point D to complete $\square ABCD$

ABCD is
 \square \square m



Because the opposite sides of \square \square m have same dirn & same length

$$\overrightarrow{BC} = \overrightarrow{DA} = \underline{q} \quad \& \quad \overrightarrow{CA} = \overrightarrow{BD} = \underline{b}$$

Thus

$$\underline{c} = \underline{b} + \underline{q} = \underline{q} + \underline{b} \quad \square \square \square$$

Be sure you really can see that \overrightarrow{BC} & \overrightarrow{DA} really represent the same vector.

3.4 Subtraction of Displacements

Displacement from B to C (\vec{BC}) is NOT the same as displacement from C to B (\vec{CB})

In fact $B \rightarrow C \rightarrow B$ is the same as doing nothing; hence if

$$\underline{a} \text{ is } B \rightarrow C \\ -\underline{a} \text{ is } C \rightarrow B \quad [\text{same length}] \\ \text{Opposite dirn}]$$

$$\& \underline{a} - \underline{a} = \underline{0}$$

where $\underline{0}$ is the null vector having zero length and unspecified direction.

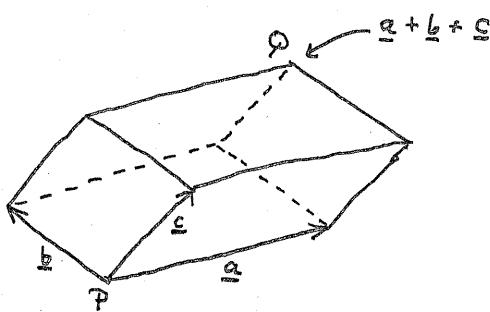
$\underline{0}$ behaves like 0 in ordinary arithmetic. [N.B. there is only a single unique $\underline{0}$, not lots with different directions]

3.5 ASSOCIATIVE LAW

Easy to see

$$(\underline{a} + \underline{b}) + \underline{c} = \underline{a} + (\underline{b} + \underline{c}) \equiv \underline{a} + \underline{b} + \underline{c}$$

from geometry of parallelepiped.



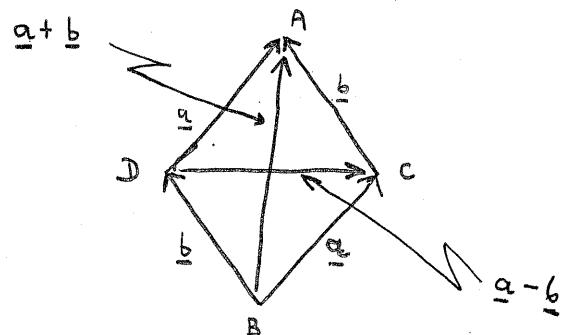
Starting at P we finish at Q by adding \underline{a} , \underline{b} & \underline{c} in any order. Every face of the drawing is a 119^{th} .

What about DISTRIBUTIVE LAW?

> First need to define multiplication of vectors by numbers.

Thus get

$$\vec{DC} = \underline{a} - \underline{b} \text{ in our figure}$$



This is because

$$\vec{DA} + \vec{AC} = \vec{DC} \\ \underline{a} - \underline{b} = \vec{DC}$$

Also

$$\underline{b} + (\underline{a} - \underline{b}) = \underline{a} + \underline{0} = \underline{a}$$

As expected.

Learn & Remember
This Figure

Put mixed scalars & vectors

3.6 Multiplication of Vector by Scalar

We multiply a vector (say \underline{a}) by a number (say 2.5) by making a new vector we call \underline{b} which is 11th to \underline{a} and has length $2.5 \times |\underline{a}|$

i.e. if $\underline{b} = 2.5 \underline{a}$

$$\text{then } |\underline{b}| = 2.5 |\underline{a}| \quad \& \quad \underline{b} \text{ is } 11^{\text{th}} \text{ to } \underline{a}$$

$$\underline{b} = \underline{a}$$

In this context the numerical constant is called a SCALAR and is often written with a lower case greek letter

$$|\lambda \underline{a}| = \lambda |\underline{a}|$$

Learn

If $\lambda < 0$ then the length is scaled by $-|\lambda|$ (or $|\lambda|!$) and the direction is flipped.

Thus $-\underline{a} = -\underline{a}$ consistent with before

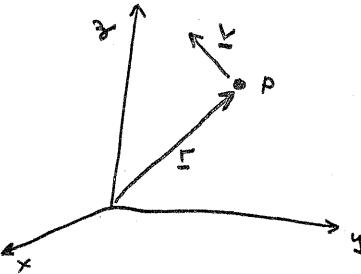
Now easy to see

$$\lambda(\underline{a} + \underline{b}) = \lambda \underline{a} + \lambda \underline{b}$$

3.4 Examples of Vectors

3.12

• Velocity



Suppose the space point P is moving. Then position vector \underline{r} is a function of time

$$\text{i.e. } \underline{r} = \underline{r}(t)$$

Learn this notation.

Define

$$\begin{aligned}\underline{v} &= \lim_{\Delta t \rightarrow 0} \frac{\underline{r}(t+\Delta t) - \underline{r}(t)}{\Delta t} \\ &= \frac{d\underline{r}}{dt} \\ &= \dot{\underline{r}} \quad (\text{"r dot"})\end{aligned}$$

does not have to be \perp to \underline{r}

Clearly a vector!

$$|\underline{v}| = \text{speed}$$

$$\boxed{\underline{v} = \text{velocity} \quad (\text{has direction})}$$

Be very careful to distinguish between speed & velocity or you will lose marks in physics exams.

[Next term you may see the lecturers use \underline{x} instead this is because they are mathematicians not physicists/chemists ...]

Position vectors are just a type of displacement vector so we know how to add etc...

3.14

• Acceleration

Acceleration, \underline{a} , is rate of change of velocity:

$$\underline{a} = \frac{d\underline{v}}{dt} = \dot{\dot{\underline{r}}} = \ddot{\underline{r}}$$

N.B. in 1-dim \underline{a} must be \perp to \underline{v} (or anti- \perp). Not so in 3D (or 2D)
⇒ curved trajectories

• Force

If \underline{F} is a force then Newton's 2nd law is:

$$\boxed{\underline{F} = m\underline{a}}$$

m = mass, a scalar.

Calculus works fine:

example $\underline{F} = \text{constant}$

$$\underline{F} = m\underline{a} = m \frac{d^2\underline{r}}{dt^2} \equiv m \ddot{\underline{r}}$$

$$\ddot{\underline{r}} = \frac{1}{m}\underline{F} \quad \text{rhs is constant}$$

$$\therefore \dot{\underline{r}} = \frac{1}{m}\underline{F} + \underline{U}_0$$

$$\boxed{\underline{r} = \frac{1}{2m}\underline{F}t^2 + \underline{U}_0t + \underline{r}_0} *$$

where $\underline{r} = \underline{r}_0$ at $t=0$
and $\dot{\underline{r}} = \underline{U}_0$ at $t=0$

We have done a (simple) 3D problem just as easily as if it were 1D! N.B.

- \underline{F} is constant but does not have to be parallel to \underline{U}_0 .
- Each term in (*) has same units.
- Not everything in physics is a vector. E.g. mass, m , is a scalar. (Much) later you will also meet TENSORS, SPINORS ...

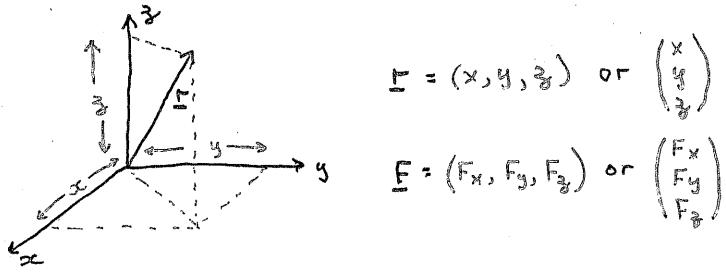
The vector equation means both sides have equal magnitudes and are parallel.

Newton's 2nd Law is a VECTOR LAW.

This is a physical law expressed in terms of vectors

3.8 Newton's Law in Components

3.16



$$\underline{F} = (x, y, z) \text{ or } \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\underline{F} = (F_x, F_y, F_z) \text{ or } \begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix}$$

We will do this in more detail in Chapter 6 - but the components of a vector are just the lengths of the projections along the axes.

$$m\ddot{x} = F_x \rightarrow \dot{x} = \frac{F_x}{m} t + P \quad P = \text{arb const.} \quad (\dot{x} \text{ at } t=0)$$

$$\rightarrow x = \frac{F_x}{2m} t^2 + Pt + Q \quad (Q = x \text{ at } t=0)$$

$$m\ddot{y} = F_y \rightarrow \dot{y} = \frac{F_y}{m} t + R \quad R = \text{arb const.}$$

$$m\ddot{z} = F_z \rightarrow \dot{z} = \frac{F_z}{m} t + S \quad S = \text{arb const.}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{t^2}{2m} \begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix} + t \begin{pmatrix} P \\ R \\ S \end{pmatrix} + \begin{pmatrix} Q \\ S \\ C \end{pmatrix}$$

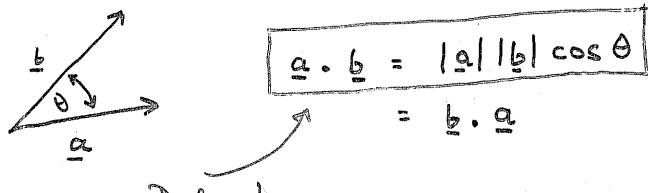
$$\underline{r} = \frac{t^2}{2m} \underline{F} + t \underline{u}_0 + \underline{r}_0$$

4.1 Scalar & Vector Products.

4.1

We have already seen how to multiply a vector by a scalar. We will now define 2 other kinds of product.

4.1.1 Scalar product (or dot product).



Definition.

N.B.

- (i) The result is a **NUMBER** = SCALAR
- (ii) $\cos(2\pi - \theta) = \cos(\theta) \therefore$ doesn't matter how we choose θ
- (iii) if $\theta > \frac{\pi}{2}$ get -ve answer.

**LEARN THIS DEFINITION
AS WELL AS YOU KNOW
YOUR NAME - NEVER FORGET IT**

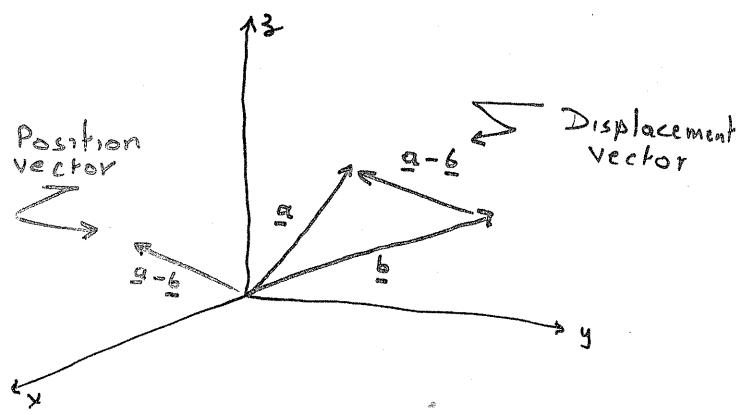
3.9 Special Case

3.1

It can be easy to forget the difference between position and displacement vectors. Both measure length.

Displacement vectors are defined completely by their 3 components

Position vectors are defined by their 3 components AND then additional information about where the origin is.



Is it $\underline{a} \cdot \underline{b}$ or $\underline{a} \cdot \underline{b}$?

Actually you see both in printed books & exam papers. I will probably not be consistent. In practice there is NEVER any confusion - if you see a dot between 2 vectors it will always mean scalar product.

Examples

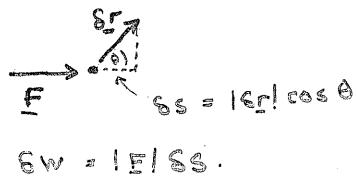
- (i) $\underline{a} \cdot \underline{a} = |\underline{a}| |\underline{a}| \cos 0^\circ = |\underline{a}|^2$ ($\cos 0^\circ = 1$)
- (ii) $\underline{a} \cdot \underline{b} = 0 \Rightarrow \underline{a} \perp \underline{b} \quad \theta = 90^\circ$
or $|\underline{a}| = 0$ } ie \underline{a} or \underline{b}
or $|\underline{b}| = 0$ } null vector
- (iii) $\underline{a} \cdot \hat{n}$ is the projection of \underline{a} in the \hat{n} direction. (defn?)



$$|\underline{a}| \cos \theta = \underline{a} \cdot \hat{n} = |\underline{a}| \hat{n}$$

4.3
iv) If a force \underline{F} acts on a body and displaces it a small distance \underline{s}_F then the work done is:

$$W_F = \underline{F} \cdot \underline{s}_F$$



Thus if $\underline{F} \perp \underline{s}_F$ no work is done - eg circular motion; orbits electrons in mag field etc

v) Potential energy of magnetic dipole (bar magnet) in magnetic field \underline{B} is

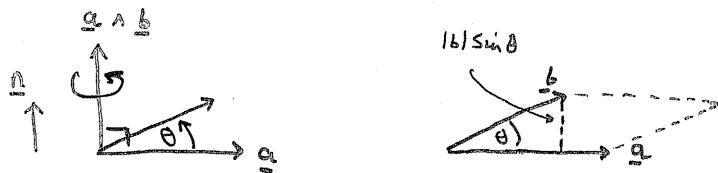
$$U = -\underline{\mu} \cdot \underline{B} \quad \text{No proof here.}$$

N.B. This slide is non-examinable in maths - it anticipates results in Physics courses.

4.2 The Vector Product

4.5

The vector product of \underline{a} & \underline{b} is $\underline{a} \wedge \underline{b}$ and has magnitude = area of parallelogram defined by \underline{a} & \underline{b} and direction \perp to plane of \underline{a} & \underline{b} in dirn of RH screw turning from \underline{a} to \underline{b}



$$\underline{a} \wedge \underline{b} = |\underline{a}| |\underline{b}| \sin \theta \hat{n} = -\underline{b} \wedge \underline{a}$$

\hat{n} unit vector \perp to plane of \underline{a} & \underline{b} in RHS Rule sense rotating from \underline{a} to \underline{b}

$\underline{a} \wedge \underline{b}$ is a vector

Write $\underline{a} \wedge \underline{b}$ or $\underline{a} \times \underline{b}$ & talk about cross product, vector product or wedge product (notation is never ambiguous)

Remark

Unlike addition of vectors we can (and do) form products between different types of vector.

Algebra

Easy to see that all the expected operations work:

$$\underline{a} \cdot \underline{b} = \underline{b} \cdot \underline{a} \quad \text{from def'}$$

$$\underline{a} \cdot (\underline{b} + \underline{c}) = \underline{a} \cdot \underline{b} + \underline{a} \cdot \underline{c} \quad (\text{proof later})$$

$$(\lambda \underline{a}) \cdot \underline{b} = \lambda (\underline{a} \cdot \underline{b}) \quad \begin{array}{l} \text{even if } \lambda < 0 \\ \text{think about } \theta. \end{array}$$

Examples

$$(i) \text{ If } \underline{a} \parallel \underline{b} \text{ then } \theta = 0 \text{ hence } \sin \theta = 0$$

$$\Rightarrow \underline{a} \wedge \underline{b} = 0$$

(just as well \parallel vectors don't define a plane).

$$\therefore \underline{a} \wedge \underline{b} = 0 \Rightarrow \underline{a} \& \underline{b} \parallel \text{ (or anti-parallel)}$$

or $|\underline{a}| = 0$ { trivial case
or $|\underline{b}| = 0$

$$(ii) \underline{a} \wedge \underline{a} = 0 \quad \forall \underline{a} \quad \text{LEAP}$$

$$(iii) \underline{a} \wedge \underline{b} = -\underline{b} \wedge \underline{a} \quad \text{From RHS rule}$$

$$(iv) \underline{a} \cdot (\underline{a} \wedge \underline{b}) = 0 \quad \underline{a} \wedge \underline{b} \text{ is } \perp \text{ to } \underline{a}$$

$$\underline{b} \cdot (\underline{a} \wedge \underline{b}) = 0 \quad \underline{a} \wedge \underline{b} \text{ is } \perp \text{ to } \underline{b}$$

(v) The fact that we can find a unique direction \perp to a plane is a special property of 3D space. Thus the vector product does not generalize to n dimensions, but the scalar product...

4.3 Scalar triple Product

Algebra

$$\underline{a} \wedge (\underline{b} + \underline{c}) = \underline{a} \wedge \underline{b} + \underline{a} \wedge \underline{c} \quad *$$

$$(\lambda \underline{a}) \wedge \underline{b} = \lambda (\underline{a} \wedge \underline{b})$$

But

$$\underline{a} \wedge \underline{b} = -\underline{b} \wedge \underline{a}$$

$$\& \underline{a} \wedge (\underline{b} \wedge \underline{c}) \neq (\underline{a} \wedge \underline{b}) \wedge \underline{c} \quad *$$

- VECTOR PRODUCTS ANTI-COMMUTE
- VECTOR PRODUCTS NOT ASSOCIATIVE

Otherwise [all] the algebra associated with vectors follows same rules as numbers.

[Beware some things dont make sense]

eg $\underline{a} \cdot \underline{b} \cdot \underline{c}$ does not mean anything]

wrong

$\underline{a} \cdot \underline{b} \wedge \underline{c}$
probably OK.

* Proof's Later

Not really new - but useful

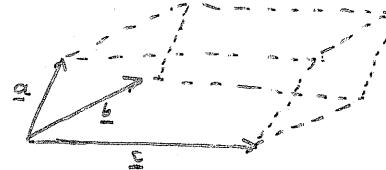
$$[\underline{a}, \underline{b}, \underline{c}] \equiv \underline{a} \cdot (\underline{b} \wedge \underline{c}) \text{ defn}$$

$$= \underline{c} \cdot (\underline{a} \wedge \underline{b}) = [\underline{c}, \underline{a}, \underline{b}]$$

$$= \underline{b} \cdot (\underline{c} \wedge \underline{a}) = [\underline{b}, \underline{c}, \underline{a}]$$

$\begin{matrix} \swarrow \underline{a} \\ \circlearrowleft \underline{c} \\ \curvearrowright \underline{b} \end{matrix}$

= volume of parallelepiped defined by $\underline{a}, \underline{b}, \underline{c}$



(Proof - exercise for you)

cyclic perm of $\underline{a}, \underline{b}, \underline{c} \Rightarrow$ no change
anti-cyclic perm \Rightarrow minus sign

$$[\underline{b} \underline{a} \underline{c}] = -[\underline{a} \underline{b} \underline{c}]$$

Aside

If this reminds you of determinants it's because it is a determinant!

4.4 Vector triple Product

$$\underline{a} \wedge (\underline{b} \wedge \underline{c})$$

Not new, remember useful formula:

$$\underline{a} \wedge (\underline{b} \wedge \underline{c}) = (\underline{a} \cdot \underline{c}) \underline{b} - (\underline{a} \cdot \underline{b}) \underline{c}$$

Hint "the middle vector gets the + sign"

Note

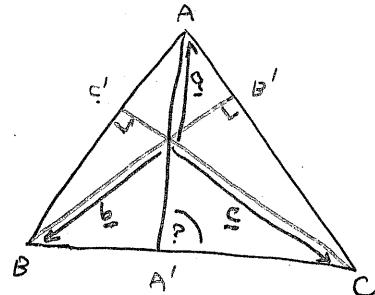
$$\begin{aligned} (\underline{a} \wedge \underline{b}) \wedge \underline{c} &= -\underline{c} \wedge (\underline{a} \wedge \underline{b}) \\ &= -(\underline{c} \cdot \underline{b}) \underline{a} + (\underline{c} \cdot \underline{a}) \underline{b} \\ &= (\underline{a} \cdot \underline{c}) \underline{b} - (\underline{c} \cdot \underline{b}) \underline{a} \\ &\neq \underline{a} \wedge (\underline{b} \wedge \underline{c}) \end{aligned}$$

Hint still works

* proof later

5. Geometry with vectors - lines & planes

5.1 Typical triangle theorem



"Altitudes of a Δ meet"

"If $\overrightarrow{BB'} \perp \overrightarrow{AC}$
& $\overrightarrow{CC'} \perp \overrightarrow{AB}$
Then $\overrightarrow{AA'} \perp \overrightarrow{BC}$ "

construct vectors $\underline{a}, \underline{b}, \underline{c}$ from crossing point to A, B, C.

$$\overrightarrow{BB'} \perp \overrightarrow{AC} \Rightarrow (\underline{c} - \underline{a}) \cdot \underline{b} = 0$$

$$\underline{c} \cdot \underline{b} = \underline{a} \cdot \underline{b}$$

$$\overrightarrow{CC'} \perp \overrightarrow{AB} \Rightarrow (\underline{b} - \underline{a}) \cdot \underline{c} = 0$$

$$\underline{b} \cdot \underline{c} = \underline{a} \cdot \underline{c}$$

$$\therefore \underline{a} \cdot \underline{b} = \underline{a} \cdot \underline{c}$$

$$\therefore (\underline{c} - \underline{b}) \cdot \underline{a} = 0$$

$$\therefore \overrightarrow{BC} \perp \overrightarrow{AA'} \quad \text{QED}$$

5.2 Equations for st. Line

5.2

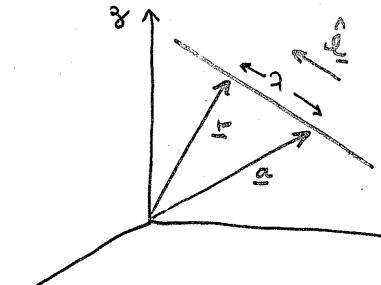
If \underline{r} is position vector & λ is a freely varying parameter, then

$$\underline{r} = \lambda \hat{\underline{e}}$$

is a st. Line \parallel_{el} to $\hat{\underline{e}}$ & going through origin.

$$\underline{r} = \underline{a} + \lambda \hat{\underline{e}}$$

st line \parallel_{el} to $\hat{\underline{e}}$ & going through \underline{a}

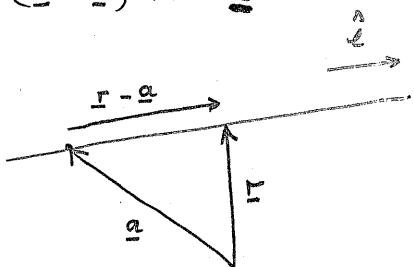


N.B. For $\hat{\underline{e}}$ = unit vector λ is distance. We can use non unit vectors to specify direction - in this case units of λ change.

Form

$$\underline{r} \wedge \hat{\underline{e}} = \underline{a} \wedge \hat{\underline{e}}$$

$$= (\underline{r} - \underline{a}) \wedge \hat{\underline{e}} = 0$$



iff \underline{r} is on line $\underline{r} - \underline{a} \parallel_{el} \hat{\underline{e}}$

$$\text{hence } (\underline{r} - \underline{a}) \wedge \hat{\underline{e}} = 0$$

• Line through 2 points $\underline{a}, \underline{b}$

\rightarrow any 2 points define a st line and $\underline{b} - \underline{a}$ is \parallel_{el} to this line

$$\underline{r} = \underline{a} + \lambda (\underline{b} - \underline{a})$$

is st line through \underline{a} & \underline{b}

$$(use (\underline{b} - \underline{a}) = \frac{\underline{b} - \underline{a}}{|\underline{b} - \underline{a}|} \text{ if want } \lambda \text{ to be distance.})$$

• Can use vector product to get rid of λ .

$$\underline{r} = \underline{a} + \lambda \hat{\underline{e}}$$

$$\therefore \underline{r} \wedge \hat{\underline{e}} = \underline{a} \wedge \hat{\underline{e}} + \lambda \hat{\underline{e}} \wedge \hat{\underline{e}}$$

$$\underline{r} \wedge \hat{\underline{e}} = \underline{a} \wedge \hat{\underline{e}}$$

RHS might be any vector \perp to $\hat{\underline{e}}$

Alternate form of equation for line through \underline{a} \parallel_{el} to $\hat{\underline{e}}$

Learn these eqns

5.3 Equations of planes

5.1

• All possible 'linear combinations' of 2 non-parallel vectors defn a plane. I.e. if \underline{p} & \underline{q} are any 2 non- \parallel_{el} vectors

$$\underline{r} = \lambda \underline{p} + \mu \underline{q}$$

defines a plane as λ & μ take all possible values.

" \underline{p} & \underline{q} span a 2D \perp subspace (plane) in 3D space."

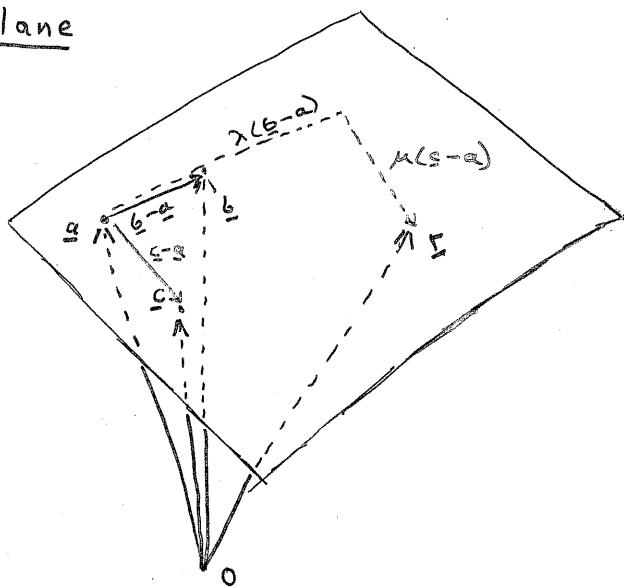
$$\underline{r} = \underline{a} + \lambda \hat{\underline{p}} + \mu \hat{\underline{q}}$$

This is either defn or phys

plane containing point \underline{a} & lines $\hat{\underline{p}}$ & $\hat{\underline{q}}$

$$\underline{r} = \underline{a} + \lambda (\underline{b} - \underline{a}) + \mu (\underline{c} - \underline{a})$$

plane through 3 points $\underline{a}, \underline{b}, \underline{c}$

Plane

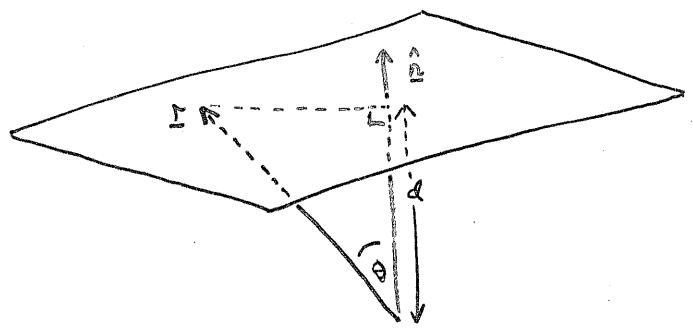
$$\underline{r} = \underline{a} + \lambda(\underline{b} - \underline{a}) + \mu(\underline{c} - \underline{a})$$

Take cross product with normal

$$\underline{n} \propto (\underline{b} - \underline{a}) \wedge (\underline{c} - \underline{a})$$

$$\underline{r} \cdot \underline{n} = \underline{a} \cdot \underline{n} = d \quad \leftarrow \text{most useful form.}$$

d is \perp^r distance of plane from origin.

Plane:

d is \perp^r dist of plane from origin

$$|\underline{r}| \cos \theta = d, \text{ for any } \underline{r} \text{ in plane}$$

$$\therefore \underline{r} \cdot \underline{n} = d$$

(LEARN)

$$\& d = \underline{a} \cdot \underline{n} \text{ for any known point } \underline{a} \text{ in the plane.}$$

5.4 Example - common point of 3 planes

5.7

Common point of $\underline{r} \cdot \underline{a} = l$, $\underline{r} \cdot \underline{b} = m$, $\underline{r} \cdot \underline{c} = n$

$\underline{a} \wedge \underline{b} \perp^r$ to normals of \underline{a} & \underline{b} planes \therefore \perp^r to common line. Consider

$$\underline{r} \wedge (\underline{a} \wedge \underline{b}) = \underline{a}(\underline{r} \cdot \underline{b}) - \underline{b}(\underline{r} \cdot \underline{a})$$

$$\therefore \underline{r} \wedge (\underline{a} \wedge \underline{b}) = ma - lb \quad \text{indep of } \underline{r}$$

$$\text{St line of form } \underline{r} \wedge \underline{n} = p \wedge \underline{n}$$

Next take $\underline{c} \wedge \underline{n}$

$$\begin{aligned} \underline{c} \wedge (\underline{r} \wedge (\underline{a} \wedge \underline{b})) &= \underline{r} \cdot \underline{c} \cdot (\underline{a} \wedge \underline{b}) - (\underline{a} \wedge \underline{b}) \cdot \underline{c} \cdot \underline{r} \\ &= \underline{r} \cdot [a, b, c] - n(\underline{a} \wedge \underline{b}) \end{aligned}$$

$$\underline{c} \wedge (ma - lb) = m \underline{c} \wedge \underline{a} - l \underline{c} \wedge \underline{b}$$

Hence

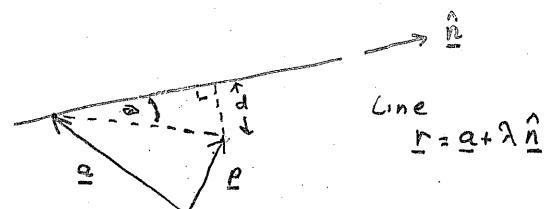
$$\boxed{\underline{r} = l \underline{b} \wedge \underline{c} + m \underline{c} \wedge \underline{a} + n \underline{a} \wedge \underline{b}} \quad [a, b, c]$$

* Can see by inspection this works

* $[a, b, c] \neq 0$ is condition for 3 normals to be non coplanar. (Toblerone bar)

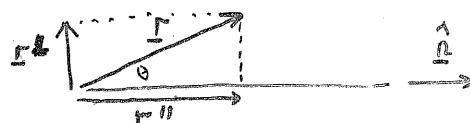
Example Distance of Point from Line

5.8



$$d = |\underline{P} - \underline{a}| \sin \theta$$

$$d = |(\underline{P} - \underline{a}) \wedge \underline{n}| \quad \text{Defn of } \wedge \text{ prod}$$

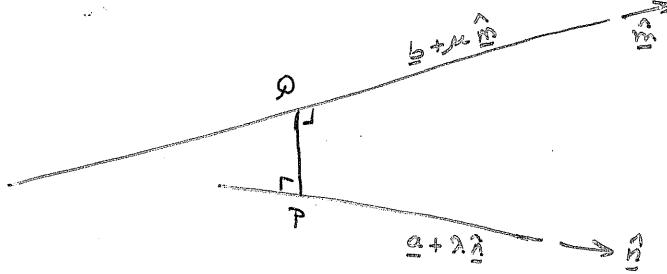
Example components of vector \parallel & \perp to \underline{n} 

$$|\underline{r}^\parallel| = r \cos \theta = \underline{r} \cdot \underline{n} \quad (\text{number})$$

$$\therefore \underline{r}^\parallel = (\underline{r} \cdot \underline{n}) \underline{n} \quad \text{vector } \parallel \wedge \underline{n}$$

$$\therefore \underline{r}^\perp = \underline{r} - \underline{r}^\parallel$$

$$\boxed{\underline{r}^\perp = \underline{r} - (\underline{r} \cdot \underline{n}) \underline{n}}$$

Example: Minimum Dist Between Skew Lines

Skew Lines $\vec{r} = \underline{a} + \lambda \underline{n}$
 $\vec{r} = \underline{b} + \mu \underline{m}$

The shortest dist, \vec{PQ} , is \perp^r to both

$$\vec{PQ} = \underline{b} + \mu \underline{m} - \underline{a} - \lambda \underline{n} \quad (\equiv \underline{Q} - \underline{P})$$

for some $\mu = \mu_P$ & $\lambda = \lambda_P$

BUT THIS VECTOR IS \perp^r to \underline{m} & \underline{n} : parallel to $\underline{n} \wedge \underline{m}$

$$\begin{aligned} \therefore |\vec{PQ}| &= \vec{PQ} \cdot \frac{\underline{n} \wedge \underline{m}}{|\underline{n} \wedge \underline{m}|} \quad \text{unit vector} \\ &= (\underline{b} - \underline{a}) \cdot \frac{(\underline{m} \wedge \underline{n})}{|\underline{m} \wedge \underline{n}|} \quad (\mu_P \text{ & } \lambda_P \\ &\quad \text{terms are zero}) \end{aligned}$$

6 Components6.1 Basis of vector space

[Easter term topic]

Given any 3 non coplanar vectors, $\{\underline{a}, \underline{b}, \underline{c}\}$, all vectors can be written as

$$\underline{r} = \lambda \underline{a} + \mu \underline{b} + \nu \underline{c}$$

for some unique $\{\lambda, \mu, \nu\}$ which depends on \underline{r} & the BASIS vectors $\{\underline{a}, \underline{b}, \underline{c}\}$.

- The basis vectors are said to SPAN the vectors space.
- $\{\lambda, \mu, \nu\}$ are the COMPONENTS of \underline{r} wrt this basis.
- This is the starting point for general treatment of vector spaces.
- A KEY THEOREM which is easy to prove is that if one basis has 3 (N) members ALL BASES have 3 (N) members. \Rightarrow gives meaning to dimension of vector space.

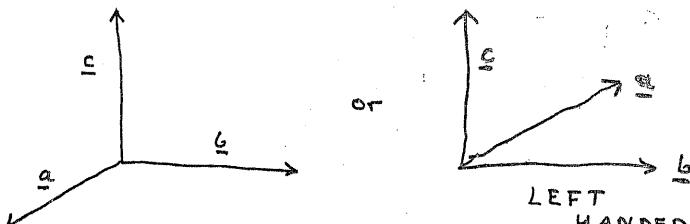
6.2

6.2

6.2 Orthogonal Bases

Specialize to mutually \perp^r basis vectors

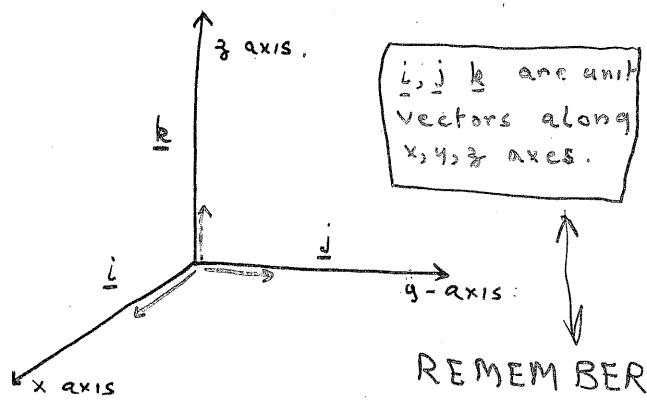
i.e. $\underline{a} \cdot \underline{b} = 0$ $\underline{b} \cdot \underline{c} = 0$ $\underline{c} \cdot \underline{a} = 0$



In fact go straight to

Right handed Orthonormal Basis

$$|\underline{a}| = |\underline{b}| = |\underline{c}| = 1 \quad \underline{a} \wedge \underline{b} = \underline{c} \text{ etc.}$$



KNOW
T USE
MEM

$i \cdot i = j \cdot j = k \cdot k = 1$	
$i \cdot j = j \cdot k = k \cdot i = 0$	
$i \wedge j = k$	$j \wedge i = -k$
$j \wedge k = i$	$k \wedge j = -i$
$k \wedge i = j$	$i \wedge k = -j$
cyclic	
anti-cyclic.	

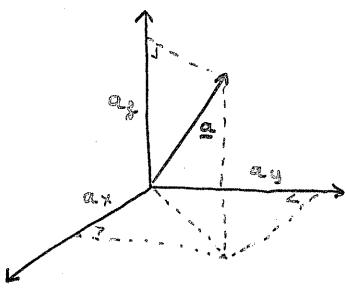
RIGHT HANDED AXES.

Aside:

N.B. The notation $\{i, j, k\}$ is pretty standard in your courses this year. However one might use $\{\hat{u}_x, \hat{u}_y, \hat{u}_z\}$ or $\{\hat{u}_1, \hat{u}_2, \hat{u}_3\}$ etc. This is less confusing when introduce a_{ij} as element of matrix & with $1, 2, 3$ not x, y, z , one can easily generalize to N dimensions.

6.3 Components of vectors

6.3



Given vector \underline{a}
& set of axes
the components
of \underline{a} are
(a_x, a_y, a_z) & are
The projections of
 \underline{a} onto the axes.

$$\underline{a} = a_x \underline{i} + a_y \underline{j} + a_z \underline{k}$$

$$a_x = \underline{a} \cdot \underline{i}$$

(by dotting with \underline{i})

$$a_y = \underline{a} \cdot \underline{j}$$

(" \underline{j})

$$a_z = \underline{a} \cdot \underline{k}$$

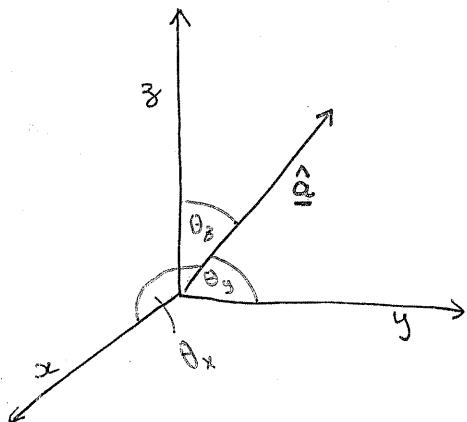
(" \underline{k})

For position vector \underline{r} usually write

$$\underline{r} = x \underline{i} + y \underline{j} + z \underline{k}$$

$$\text{or } \underline{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ or } (x, y, z)$$

Direction Cosines

6.4₂

If $\hat{\underline{a}}$ is a unit vector

$$\hat{\underline{a}} = (\hat{a}_x, \hat{a}_y, \hat{a}_z)$$

$$\text{we have } \hat{\underline{a}} \cdot \underline{i} = \hat{a}_x = \cos \theta_x \quad |\hat{\underline{a}}| = 1 \\ = \cos \theta_x$$

etc

$$\hat{\underline{a}} = (\cos \theta_x, \cos \theta_y, \cos \theta_z)$$

Called direction cosines.

For other vectors, eg, \underline{a} we
write:

$$\underline{a} = a_x \underline{i} + a_y \underline{j} + a_z \underline{k}$$

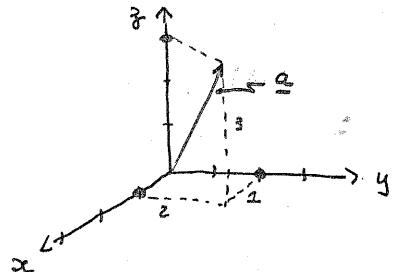
$$\text{or } \underline{a} = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} \text{ or } (a_x, a_y, a_z)$$

You will learn the difference between
'row' and 'column' vectors when
you do matrices - for now don't
worry.

Remember $\{a_x, a_y, a_z\}$ are numbers

$\therefore \underline{a} = (1, 2, 3)$ is fine this
means

$$\underline{a} = \underline{i} + 2\underline{j} + 3\underline{k}$$



Now we can obtain LOTS of useful
formulae to do calculations & prove
outstanding theorems.

6.4 SCALAR PRODUCT

$$\underline{a} = (a_x, a_y, a_z), \underline{b} = (b_x, b_y, b_z)$$

$$\underline{a} \cdot \underline{b} = (a_x \underline{i} + a_y \underline{j} + a_z \underline{k}) \cdot (b_x \underline{i} + b_y \underline{j} + b_z \underline{k})$$

$$\begin{aligned} &= a_x b_x \underline{i} \cdot \underline{i} + a_x b_y \underline{i} \cdot \underline{j} + a_x b_z \underline{i} \cdot \underline{k} \\ &\quad + a_y b_x \underline{j} \cdot \underline{i} + a_y b_y \underline{j} \cdot \underline{j} + a_y b_z \underline{j} \cdot \underline{k} \\ &\quad + a_z b_x \underline{k} \cdot \underline{i} + a_z b_y \underline{k} \cdot \underline{j} + a_z b_z \underline{k} \cdot \underline{k} \end{aligned}$$

$$\underline{a} \cdot \underline{b} = a_x b_x + a_y b_y + a_z b_z$$

MUST LEARN

Hence

$$|\underline{a}|^2 = \underline{a} \cdot \underline{a} = a_x^2 + a_y^2 + a_z^2$$

3D Pythagoras

[+ Natural extension to N dimensions]

Is invariant: $\underline{a} \cdot \underline{b} = \underline{b} \cdot \underline{a}$

It is also obvious from defn of addition that:

$$\underline{a} + \underline{b} = (a_x + b_x, a_y + b_y, a_z + b_z)$$

Add (subtract) vectors

\leftrightarrow add (subtract) components

Likewise

$$\lambda \underline{a} = (\lambda a_x, \lambda a_y, \lambda a_z)$$

Multiply vector by $\lambda \leftrightarrow$ multiply comps by λ .

Thus

$$\begin{aligned} \underline{a} \cdot (\underline{b} + \underline{c}) &= a_x(b_x + c_x) + a_y(b_y + c_y) \\ &\quad + a_z(b_z + c_z) \\ &= a_x b_x + a_y b_y + a_z b_z + a_x c_x \\ &\quad + a_y c_y + a_z c_z \\ &= \underline{a} \cdot \underline{b} + \underline{a} \cdot \underline{c} \end{aligned}$$

Proving easily the distributive law for scalar product.

6.5 VECTOR PRODUCT

$$\begin{aligned} \underline{a} \cdot \underline{b} &= (a_x \underline{i} + a_y \underline{j} + a_z \underline{k}) \cdot (b_x \underline{i} + b_y \underline{j} + b_z \underline{k}) \\ &= a_x b_x \underline{i} \cdot \underline{i} + a_x b_y \underline{i} \cdot \underline{j} + a_x b_z \underline{i} \cdot \underline{k} \\ &\quad + a_y b_x \underline{j} \cdot \underline{i} + a_y b_y \underline{j} \cdot \underline{j} + a_y b_z \underline{j} \cdot \underline{k} \\ &\quad + a_z b_x \underline{k} \cdot \underline{i} + a_z b_y \underline{k} \cdot \underline{j} + a_z b_z \underline{k} \cdot \underline{k} \\ &= a_x b_x \underline{i} - a_x b_z \underline{j} - a_y b_x \underline{k} + a_y b_y \underline{i} + a_y b_z \underline{k} \\ &\quad + a_z b_x \underline{j} - a_z b_y \underline{i} + a_z b_z \underline{o} \end{aligned}$$

$$\begin{aligned} \underline{a} \cdot \underline{b} &= (a_y b_z - a_z b_y) \underline{i} \\ &\quad + (a_z b_x - a_x b_z) \underline{j} \\ &\quad + (a_x b_y - a_y b_x) \underline{k} \end{aligned}$$

LEARN

$$= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} \text{ determinant}$$

6.6 Triple Products

6.7

$$\text{since } \underline{b} \cdot \underline{c} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}$$

$$\underline{a} \cdot (\underline{b} \cdot \underline{c}) = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} \leftarrow \begin{matrix} \text{Generalizes} \\ \text{to} \\ N \\ \text{Dimensions!} \end{matrix}$$

Hence symmetry properties

NEXT

$$\begin{aligned} \underline{a} \cdot (\underline{b} \cdot \underline{c}) &= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_x & a_y & a_z \\ b_y c_z - b_z c_y & b_z c_x - b_x c_z & b_x c_y - b_y c_x \end{vmatrix} \\ &= \underline{i} \{ a_y b_x c_y - a_y b_y c_x - a_z b_z c_x + a_z b_x c_z \} \\ &\quad + \underline{j} \{ a_z b_y c_x - a_z b_x c_y - a_x b_x c_y + a_x b_y c_x \} \\ &\quad + \underline{k} \{ a_x b_z c_x - a_x b_x c_z - a_y b_y c_z + a_y b_z c_y \} \\ &\quad + \underline{i} \{ a_x b_x c_x - a_x b_x c_x \} \\ &\quad + \underline{j} \{ a_y b_y c_y - a_y b_y c_y \} \\ &\quad + \underline{k} \{ a_z b_z c_z - a_z b_z c_z \} \end{aligned}$$

$$\begin{aligned} &= \underline{i} \{ a_x b_x c_x + a_y b_y c_y + a_z b_z c_z \} \\ &\quad - \underline{a}_x b_x c_x - \underline{a}_y b_y c_x - \underline{a}_z b_z c_x \} \\ &\quad + \underline{j} \{ a_z b_y c_x + a_y b_y c_y + a_x b_y c_x \} \\ &\quad - \underline{a}_z b_z c_y - \underline{a}_y b_y c_y - \underline{a}_x b_x c_y \} \\ &\quad + \underline{k} \{ a_x b_z c_x + a_y b_z c_y + a_z b_z c_z \} \\ &\quad - \underline{a}_x b_x c_z - \underline{a}_y b_y c_z - \underline{a}_z b_z c_z \} \end{aligned}$$

$$\begin{aligned} &= \underline{i} b_x \{ a_x c_x + a_y c_y + a_z c_z \} \\ &\quad + \underline{j} b_y \{ a_x c_x + a_y c_y + a_z c_z \} \\ &\quad + \underline{k} b_z \{ a_x c_x + a_y c_y + a_z c_z \} \\ &\quad - \underline{i} c_x \{ a_x b_x + a_y b_y + a_z b_z \} \\ &\quad - \underline{j} c_y \{ a_x b_x + a_y b_y + a_z b_z \} \\ &\quad - \underline{k} c_z \{ a_x b_x + a_y b_y + a_z b_z \} \\ &= (\underline{a} \cdot \underline{c})(b_x \underline{i} + b_y \underline{j} + b_z \underline{k}) \\ &\quad - (\underline{a} \cdot \underline{b})(c_x \underline{i} + c_y \underline{j} + c_z \underline{k}) \end{aligned}$$

$$\underline{a} \cdot (\underline{b} \cdot \underline{c}) = (\underline{a} \cdot \underline{c}) \underline{b} - (\underline{a} \cdot \underline{b}) \underline{c}$$

REMEMBER!

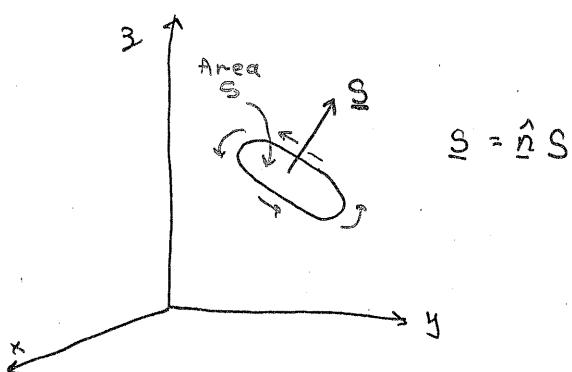
Remark

The above is fairly heroic! By the end of this year you should be able to see through a calculation like this. Actually there is a lot of symmetry so it's not really that bad.

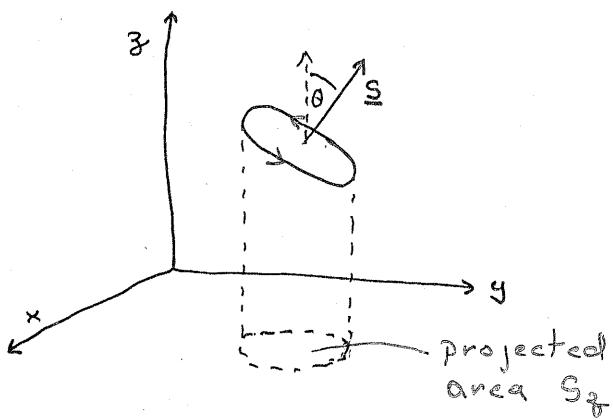
(In the Easter term you will meet summation convention & Eijk. This will give a simple elegant proof of the above - 5 short lines).

7 TWISTY VECTORS

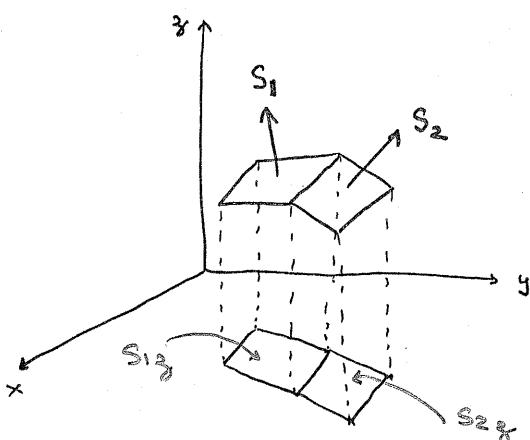
i.e. Vectors which depend on RHSR

7.1 Vector Surface Area.

Define vector surface area of finite plane surface as \underline{S} where $|S| = \text{area}$ & $\hat{\underline{S}}$ is in dirn of normal. Use RHSR to fix direction \pm (this is extra info and in practice has to do with problem in hand).

Components of \underline{S} :

- $S_z = z\text{-comp of } \underline{S} = |S| \cos \theta$
= projection of area onto x-y plane
- Similarly S_x & S_y are projections onto y-z & z-x planes.
- Projections depend only on norm not details of surface
- $\Rightarrow \underline{S}$ is a good vector because $\underline{S}_1 + \underline{S}_2$ for 2 surfaces will add in correct way.

Addition of vector surface areas

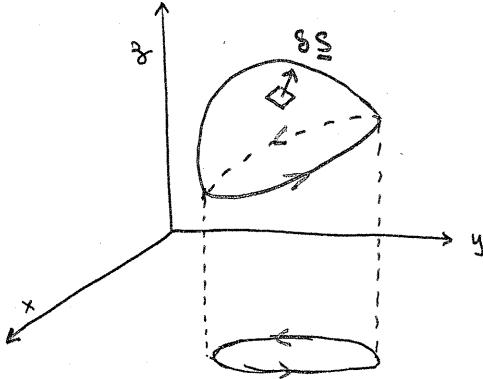
projection of total area onto x-y plane = sum of individual projections
= projected area of rim
 \therefore if $\underline{S} = \underline{S}_1 + \underline{S}_2$

$$S_{xy} = S_{1xy} + S_{2xy} \text{ etc.}$$

$$\text{Hence } \underline{S}_1 + \underline{S}_2 = \underline{S}_2 + \underline{S}_1$$

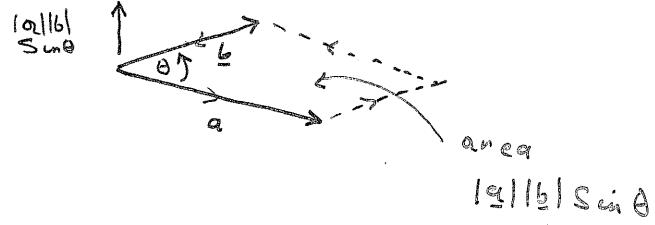
\therefore ... norm ... r

Any surface (for next term)



7.2 Parallelogram

$\underline{a} \wedge \underline{b}$ is in fact the vector area of the parallelogram defined by \underline{a} & \underline{b}



Vector area \underline{S}

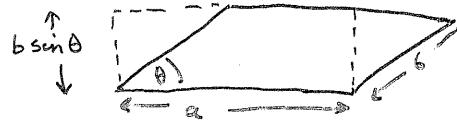
= sum of many $d\underline{S}$

$$\rightarrow \int d\underline{S}$$

over
surface

where dimⁿ of each element is out (RHSR) of surface & is \perp to local surface.

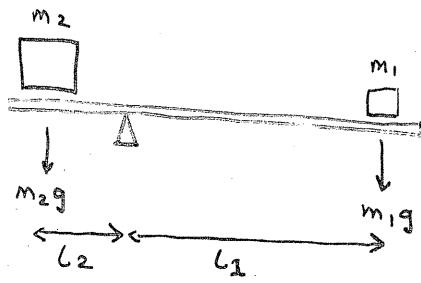
- Answer depends only on RIM.
- For closed surface $\underline{S} = \underline{0}$ (no RIM)



$$\begin{aligned} \text{Area} &= \text{Area of rectangle} \\ &= a \times b \sin \theta \\ &\text{as required.} \end{aligned}$$

7.3 Couples in mechanics

You may have met the idea of moments of forces in mechanics



To Balance

$$l_1 m_1 g = l_2 m_2 g$$

The quantity distance \times force is the moment of a force about an axis.

We make this into a vector using RHSR for sense in which force is trying to rotate object.
 \Rightarrow Couple

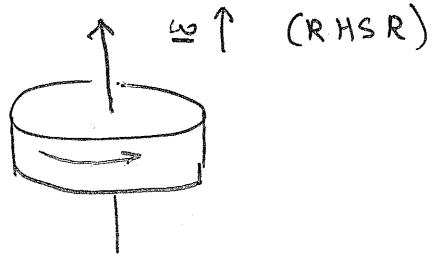
$$m_2 g \underline{l}_2 \hat{n} \rightarrow \text{out of paper}$$

$\left. \begin{array}{l} \text{add to} \\ \underline{0} \end{array} \right\}$

$$m_1 g \underline{l}_1 - \underline{n} \rightarrow \text{into paper}$$

$\left. \begin{array}{l} \text{AS.} \\ \dots \end{array} \right\}$

7.4 Angular velocity & angular momentum



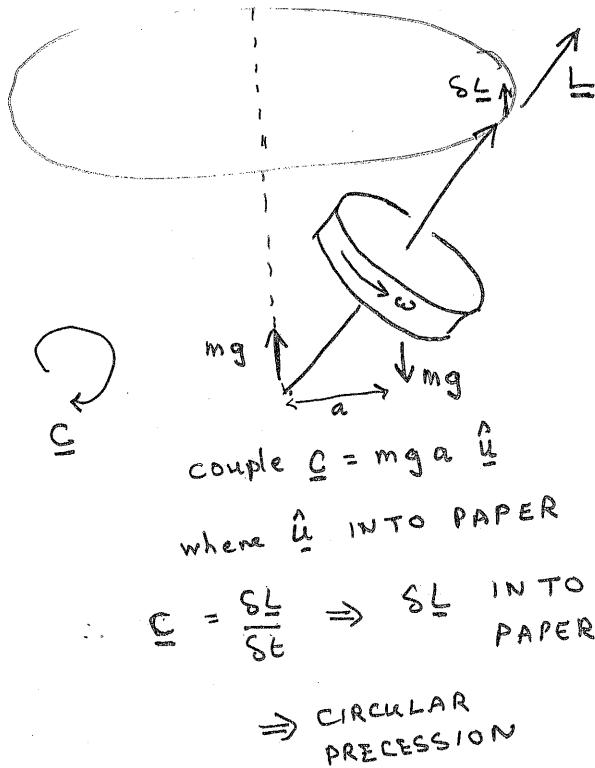
- A body rotating about an axis with angular speed ω has angular velocity $\underline{\omega}$ where $|\underline{\omega}| = \omega$ & $\underline{\omega}$ is in RHSR sense of rotation.
- Angular momentum $\underline{L} = I \underline{\omega}$ where I is moment of inertia
- $\underline{C} = \frac{d}{dt} \underline{L} = I \frac{d\underline{\omega}}{dt}$
- is twisty version of Newton's 2nd law

$$\underline{F} = \frac{d(m\underline{v})}{dt} = m \frac{d\underline{v}}{dt}$$

(This is Link with physics - NOT FOR EXAM)

Gyroscope

[NOT FOR MATHS EXAM]



CF electrons in MRI etc.

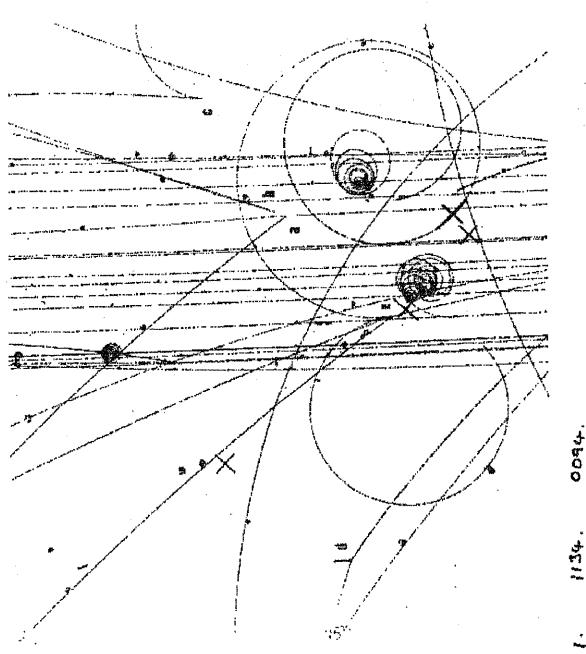
7.4

7.5 Lorentz Force in electro-magnetism

An electric charge q moving with velocity \underline{v} in a magnetic field \underline{B} feels force:

$$\boxed{\underline{F} = q \underline{v} \wedge \underline{B}}$$

- No proof here (physics course)
- $q\underline{v}$ is electric current
- Replaces all the various 'hand' rules you might have learnt
- \underline{F} is \perp to \underline{v} \therefore does no work.
- Circular motion \perp to \underline{B} is typical.



7.6 FINITE ROTATIONS

7.6

I have glossed over one important detail.

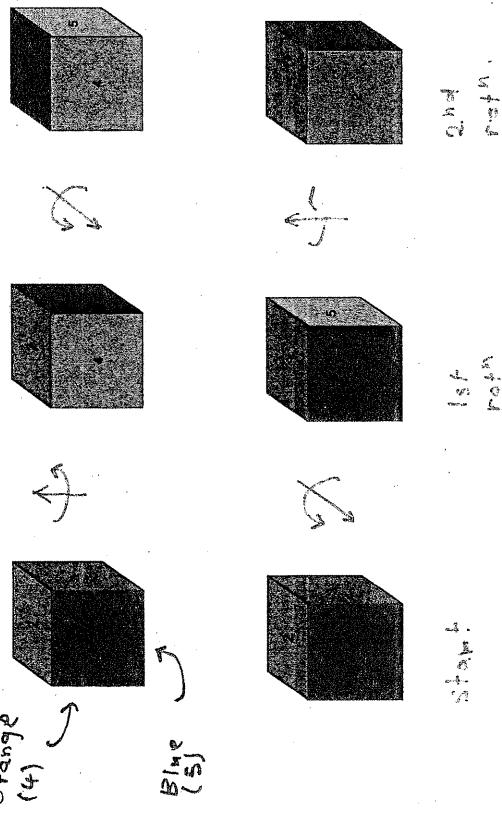
If you try to define a finite rotation as a vector using the RHSR you find this doesn't work because

finite rotations about different axes do
NOT COMMUTE
 \therefore NOT VECTORS.

IE ORDER OF ROTATIONS MATTERS
(This is why Rubik's Cube is hard)

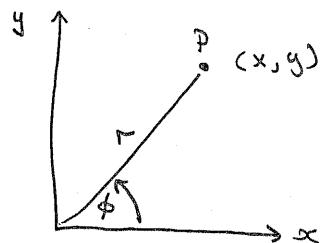
Infinitesimal rotations do commute which is why angular momentum & angular velocity are good vectors.
No time to go into detail.

Finite Rotations Don't Commute



8 Polar Coordinates

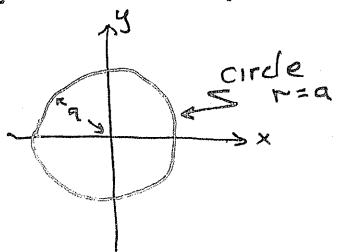
8.1 Plane Polar Coordinates (2D)



Posⁿ of P specified by (x, y) or $(r \phi)$

$$\begin{aligned} x &= r \cos \phi & r &= \sqrt{x^2+y^2} & r \geq 0 \\ y &= r \sin \phi & \phi &= \tan^{-1} \frac{y}{x} & 0 \leq \phi < 2\pi \end{aligned}$$

Useful for problems with circular sym.,
eg $r=a$ is eqⁿ for circle



$\because \frac{x}{r} = \frac{x}{r} \text{ get } \phi \text{ from } \cos \phi = \frac{x}{r} \text{ & } \sin \phi = \frac{y}{r} \text{ ATANZ}$

NB People sometimes use θ instead of ϕ . Since only one angle this is OK.

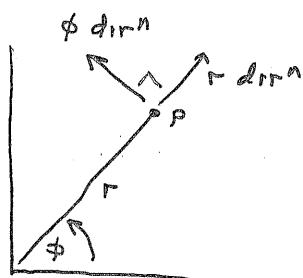
8.1

What are the r & ϕ directions?

We know what we mean by x & y dirns in cartesian coords. These are \parallel to x & y axes.

Actually the " \propto dirn" is direction we move in when x changes with y const. It just so happens that this is always the same dirn.

Likewise the " r " and " ϕ " dirns are directions we move in changing one coord & keeping other fixed.

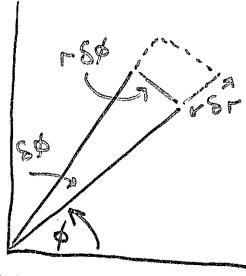


Now the coord directions are LOCAL, ie they change as P moves

But the local coord directions are always \perp ie. **locally orthogonal**
— very important

8.1 Area element

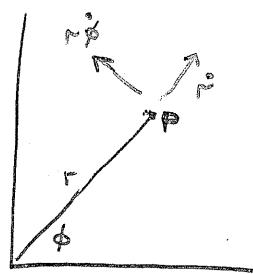
Need this when come to do 2D integrals (soon!)



$$\boxed{\text{Area element} = r dr d\phi}$$

8.2 Velocity

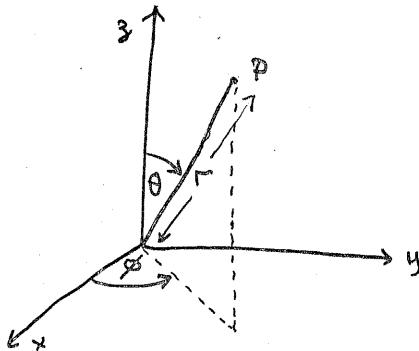
If P is moving, comps of velocity are given by



LB

8.3 Spherical Polar Coordinates

This is a not so simple extension of plane polars into 3D - add a 2nd angle.



LEARN

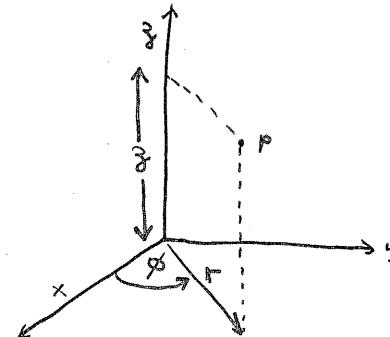
$$\begin{aligned} x &= r \sin \theta \cos \phi & 0 \leq \phi < 2\pi \\ y &= r \sin \theta \sin \phi & 0 \leq \theta \leq \pi \\ z &= r \cos \theta \end{aligned}$$

- Useful for problems with spherical sym. Eg 3D $\frac{1}{r^2}$ forces. H atom!
- The θ angle is ALWAYS the angle with the z axis. The POLAR angle.
- The ϕ angle is ALWAYS the angle of the projection of r wrt the x axis. The AZIMUTHAL angle

8.4

8.2 Cylindrical Polar Coordinates

This is a simple extension of 2D Plane Polar coordinates by adding a z axis $\perp r$ to $r\phi$ plane:



$$\begin{aligned} x &= r \cos \phi & 0 \leq \phi < 2\pi \\ y &= r \sin \phi & 0 \leq r < \infty \\ z &= z \end{aligned}$$

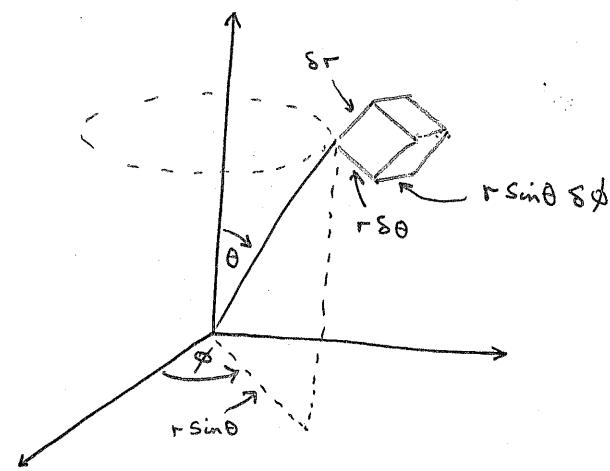
The ' z dirn' is obviously in the z dirn.
Volume element

$$\boxed{dV = r dr d\phi dz}$$

Useful for cylindrical sym.

8.5

8.3 Volume element in spherical polars

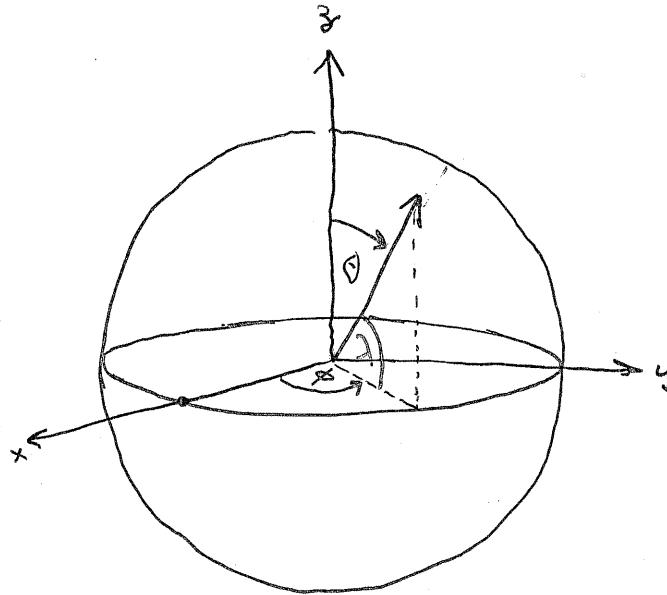


The ' r ' ' θ ' & ' ϕ ' dirns are all \perp & sweep out Δr , $\Delta \theta$, $r \sin \theta \Delta \phi$

$$\therefore \boxed{dV = r^2 \sin \theta dr d\theta d\phi} \quad \text{LEARN}$$

really important
used all the time
in Physics & Chemistry.

&
NEVER
FORGET



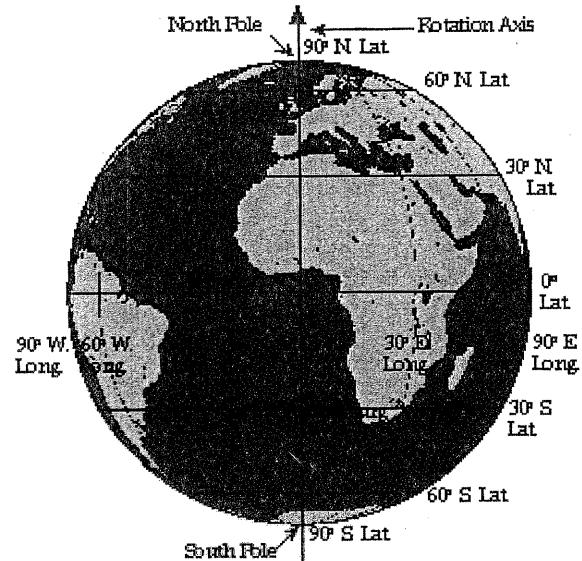
If used for navigation

ϕ is LONGITUDE

$\lambda = \frac{\pi}{2} - \theta$ is LATITUDE

$x-z$ plane is MERIDINAL PLANE

$x-y$ plane is EQUATORIAL PLANE



End of Spherical Polars for the time being. You will see many examples in your other courses - especially 3D \int 's.

14 COMPLEX NUMBERS



14.1

This topic is VERT important - it seems to be "pure maths" at first - but is in fact vital in many areas of science.

It is also easy:

- 1 Defⁿ (i)
- 1 Important theorem (Euler's Formula)
- several tricks.

14.1 Definitions

Define i as a number such that

$$i^2 = -1$$

$+i$ & $-i$ are then solutions of

$$x^2 = -1$$

just as $+1$ & -1 are solutions of

$$x^2 = 1$$

similarly if we want to solve, say,

14.2

$$x^2 - 2x + 2 = 0$$

get $x = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm \sqrt{-1}$

or $x = 1 \pm i$

Thus we can now solve ALL quadratics.

i is called an "imaginary" number
something like $1+i$ is called a
complex number.

The symbol z is usually used for
complex numbers & we write:

$$\boxed{z = x + iy}$$

where

$$x \leftrightarrow \text{real part of } z \equiv \text{Re}(z)$$

$$y \leftrightarrow \text{imaginary part of } z \equiv \text{Im}(z)$$

x & y are ordinary (real)
numbers.

② ZERO

$$z_1 = 0 \text{ if } x_1 = 0 \text{ and } y_1 = 0$$

i.e. BOTH Re & Im parts are zero.

③ ADDITION

$$z = z_1 + z_2 \text{, i.e. } x + iy = (x_1 + iy_1) + (x_2 + iy_2)$$

$$\text{then } x = x_1 + x_2 \text{ & } y = y_1 + y_2$$

$$\text{i.e. } \text{Re } z = \text{sum of Re parts}$$

$$\text{Im } z = \text{sum of Im parts}$$

$$z = z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$$

④ SUBTRACTION

$$z = z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2)$$

$$\text{i.e. } \text{Re}(z) = \text{diff of Re parts}$$

$$\text{Im}(z) = \text{diff of Im parts.}$$

(Hence

$$z_1 - z_1 = 0 \text{ etc.})$$

14.2 Complex Arithmetic

14.3

We manipulate complex numbers just like real numbers, except we replace i^2 by -1 whenever it turns up.

Complex number obey the usual rules

commutative wrt + & \times
associative wrt + & \times
distributive wrt + & \times

There are no nasty surprises.

In detail if $z_1 = x_1 + iy_1$ and
 $z_2 = x_2 + iy_2$ are 2 complex numbers:

① EQUALITY

$$z_1 = z_2 \text{ if}$$

$$x_1 = x_2 \text{ and } y_1 = y_2$$

i.e. Re Parts equal and
Im Parts equal

$$(x_1 + iy_1 = x_2 + iy_2)$$

⑤ MULTIPLICATION

14.5

work through the defⁿ

$$z = z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2)$$

$$= x_1 x_2 + ix_1 y_2 + iy_1 x_2 + i^2 y_1 y_2$$

$$= (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + y_1 x_2)$$

$$\boxed{\text{Re}(z_1 z_2) = x_1 x_2 - y_1 y_2}$$

$$\boxed{\text{Im}(z_1 z_2) = x_1 y_2 + y_1 x_2}$$

In practice you hardly need to remember this - you can always derive it as need - and there is a better way to do multiplication.
Simple examples

$$2z_1 = 2x_1 + iy_1$$

$$(\text{hence } z_1 + z_1 = 2z_1 \text{ etc.})$$

$$iz_1 = ix_1 - y_1$$

$$\therefore \text{Re}(iz_1) = -y_1$$

6 Complex Conjugate

[useful to look at this before we do division]

$$\text{if } z = x+iy$$

$z^* = x-iy$ and is the complex conjugate of z

i.e. change sign of Im Part to get complex conjugate.

Then

$$zz^* = x^2 + y^2 + i(-xy + xy)$$

$$\therefore \boxed{zz^* = x^2 + y^2} \quad \text{Real and } \geq 0$$

LEARN

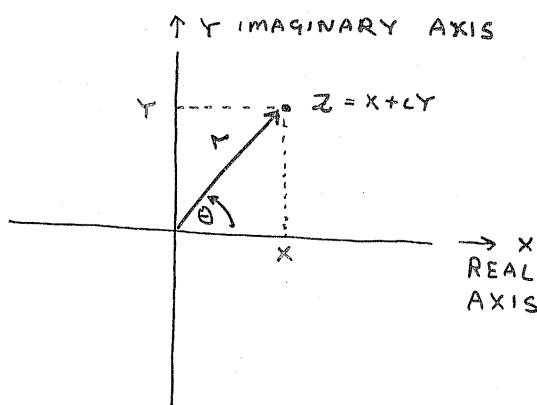
In fact $\sqrt{x^2 + y^2}$ is called the MODULUS (OR ABSOLUTE VALUE) of z and is the generalization of $|x|$ for real numbers

$$|z| \equiv \text{MOD}(z) = \sqrt{zz^*} = \sqrt{x^2 + y^2}$$

$$\text{ALSO } z+z^* = 2x, \quad z-z^* = 2iy$$

4.3 ARGAND DIAGRAM (COMPLEX PLANE)

The notation $z = x+iy$ suggests we can make a 1:1 correspondence between complex numbers and 2D position vectors $r = (x, y)$. I.E. Complex numbers represent points in a 2^d plane:



The complex number $z = x+iy$ corresponds to the point with Cartesian coordinates (x, y) or polar coordinates r, θ .

14.6

In later work if z is a complicated expression, you can get z^* by changing the sign of every i in the expression.

$$\text{eg } z = \frac{6}{1+iy} e^{ix} \cos(1+i)$$

$$\text{then } z^* = \frac{6}{1-iy} e^{-ix} \cos(1-i)$$

(we will define these functions soon)

(7) DIVISION

if $z = x+iy$ what is $\frac{1}{z}$?

$$\text{well: } \frac{1}{z} = \frac{\overline{z}}{\overline{z}\overline{z}} \cdot \frac{1}{\overline{z}} = \frac{x-iy}{x^2+y^2}$$

$$\begin{aligned} \frac{1}{z} &= \frac{x}{x^2+y^2} - i \frac{y}{x^2+y^2} \\ &= \frac{x}{|z|^2} - i \frac{y}{|z|^2} \end{aligned}$$

LEARN THE TRICK.

Hence $\frac{1}{z}$ exists, unless $|z|=0$
i.e unless $z=0$ — just like real numbers.

$$\boxed{\frac{1}{z} = -i}$$

LEARN

Notice

$$r = \sqrt{x^2 + y^2} = |z|$$

$$\text{and } \theta = \tan^{-1} \frac{y}{x} \quad (+\text{check which quadrant})$$

$$\left. \begin{array}{l} \sin \theta = \frac{y}{r} \\ \cos \theta = \frac{x}{r} \end{array} \right\} \quad \begin{array}{l} \text{Defines unique } \\ \text{ } \quad 0 \leq \theta < 2\pi \end{array}$$

Thus since

$$x = r \cos \theta$$

$$\& y = r \sin \theta, \text{ we get}$$

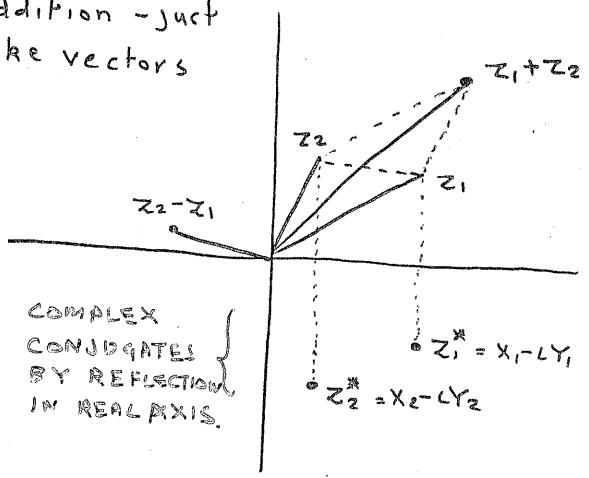
$$\boxed{z = |z|(\cos \theta + i \sin \theta)}$$

Addition & Subtraction of complex numbers follows the 11gm law for vector addition/subtraction.

The angle θ is called the phase or argument of z . written Arg(z).

14.7

Addition - just like vectors

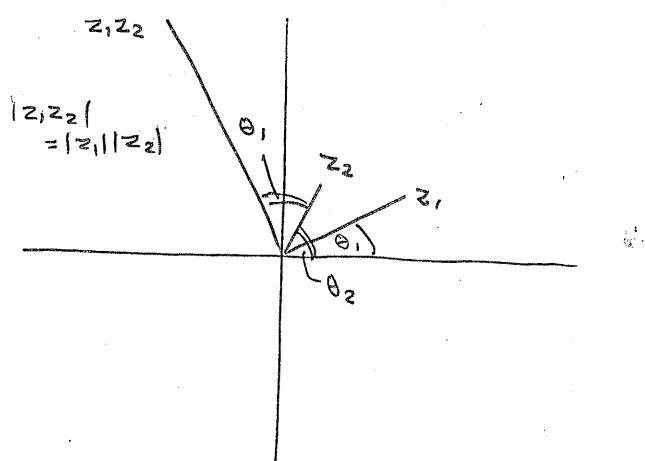


COMPLEX MULTIPLICATION ANALOGY HAS NO VECTOR
AWALOGY (NOT DOT OR CROSS PRODUCTS)
BUT:

$$\begin{aligned} z_1 z_2 &= |z_1|(\cos\theta_1 + i\sin\theta_1) |z_2|(\cos\theta_2 + i\sin\theta_2) \\ &= |z_1||z_2| \left[(\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2, \sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2) \right. \\ &\quad \left. + i (\cos\theta_1 \sin\theta_2 + \sin\theta_1 \cos\theta_2, \cos\theta_2) \right] \\ &= |z_1||z_2| (\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)) \end{aligned}$$

i.e. multiply mods & add phases

L16.



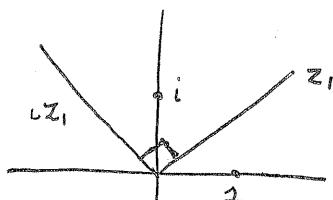
Special Cases.

if $|z_2|=1$ just rotate z_1 by angle $\theta_2 = \arg(z_2)$ to get $z_1 z_2$

if $z_2 = i$ just rotate by 90°

$$|i| = \sqrt{0+1} = 1$$

$$\arg(i) = 90^\circ$$



At the end of the last lecture

$$\text{if } z_1 = |z_1|(\cos\theta_1 + i\sin\theta_1)$$

$$\& z_2 = |z_2|(\cos\theta_2 + i\sin\theta_2)$$

$$z_1 z_2 = |z_1||z_2| \{$$

$$(\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2, \sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2)$$

$$+ i (\cos\theta_1 \sin\theta_2 + \sin\theta_1 \cos\theta_2, \cos\theta_2) \}$$

$$= |z_1||z_2| \{ \cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2) \}$$

i.e. to multiply complex numbers:

multiply moduli &
add arguments
(phases)

14.4 De Moivre's Theorem.

if $|z|=1$ & $\arg(z) = \theta$ then

$$z = \cos\theta + i\sin\theta$$

$$\text{and } z^2 = z z = \cos 2\theta + i\sin 2\theta$$

$$\text{hence } z z^2 = z^3 = \cos 3\theta + i\sin 3\theta$$

and so on, E.11

$$z^n = \cos n\theta + i\sin n\theta$$

LEARN

$$\text{or } (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta \quad \text{DeM}$$

Can get us full trig identities, e.g. $n=5$

$$\begin{aligned} \cos^5\theta + 5i\sin\theta\cos^4\theta - 10\cos^3\theta\sin^2\theta \\ - 10i\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + i\sin^5\theta \\ = \cos 5\theta + i\sin 5\theta \end{aligned}$$

Equate Re & Im parts:

$$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$$

$$\sin 5\theta = 5\sin^5\theta - 10\sin^3\theta\cos^2\theta + 5\sin\theta\cos^4\theta$$

etc.

14.5 Euler's Formula

Actually DeMoivre's Thm is a simple example of a MUCH MORE IMPORTANT result.

Define

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots$$

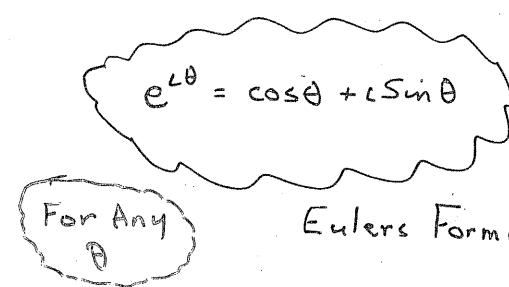
for any complex number z . This series is absolutely convergent for all z (not just all real numbers). In fact the RATIO TEST works equally for complex numbers if we replace $|x|$ by $|z|$!

$$\begin{aligned} \text{then: } e^{i\theta} &= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} - \frac{\theta^6}{6!} \dots \\ &= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} \dots \\ &\quad + i(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} \dots) \end{aligned}$$

hence ...

14.13

14.14



For Any
 θ

Euler's Formula.

This is the nicest & most important result in the whole course.

It is the key that unlocks a huge number of applications

Thus for any complex number z :

$$z = x + iy = r e^{i\theta}$$

$$\text{where } r = |z| = \sqrt{x^2 + y^2}$$

$$\text{& } \theta = \arg(z) = (\tan^{-1} \frac{y}{x})$$

Hence

$$\begin{aligned} z_1 z_2 &= (r_1 e^{i\theta_1})(r_2 e^{i\theta_2}) \\ &= r_1 r_2 e^{i(\theta_1 + \theta_2)} \end{aligned}$$

As expected
But EASY

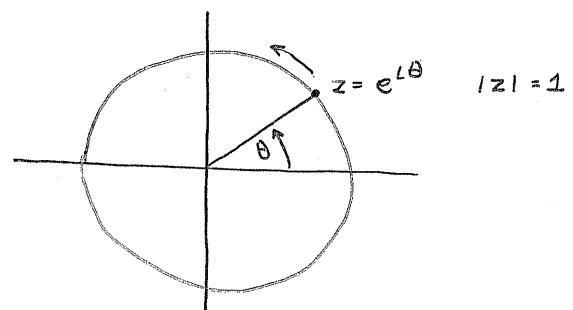
14.14₂

14.14

Applications

The n^{th} roots of unity

The locus of $z = e^{i\theta}$ is the unit circle as θ goes from $0 \rightarrow 2\pi$



If we let θ increase beyond 2π we keep going round

$$\text{thus } z = e^{i\theta}, e^{i(\theta+2\pi)}, e^{i(\theta+4\pi)} \dots$$

all represent the same point.

$$e^{2\pi i} = 1$$

Hence, putting $\theta = 0$

$$1 = e^{i0}, e^{i2\pi}, e^{i4\pi} \dots e^{i2n\pi} \dots$$

Now take $\frac{1}{n}$ 'th power - act n different ways

$\ln(-1) = \pi i$ in 1727.

One could claim that mathematical analysis began with Euler. In 1748 in *Introductio in analysin infinitorum* Euler made ideas of Johann Bernoulli more precise in defining a function, and he stated that mathematical analysis was the study of functions. This work bases the calculus on the theory of elementary functions rather than on geometric curves, as had been done previously. Also in this work Euler gave the formula

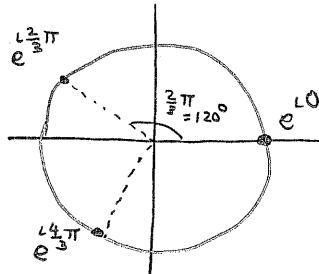
$$e^{ix} = \cos x + i \sin x.$$

In *Introductio in analysin infinitorum* Euler dealt with logarithms of a variable taking only positive values although he had discovered the formula

E.g. $n=3$,

$$1^{\frac{1}{3}} = e^{i0} \text{ or } e^{i\frac{2\pi}{3}} \text{ or } e^{i\frac{4\pi}{3}} \text{ or } e^{i\frac{6\pi}{3}} \dots$$

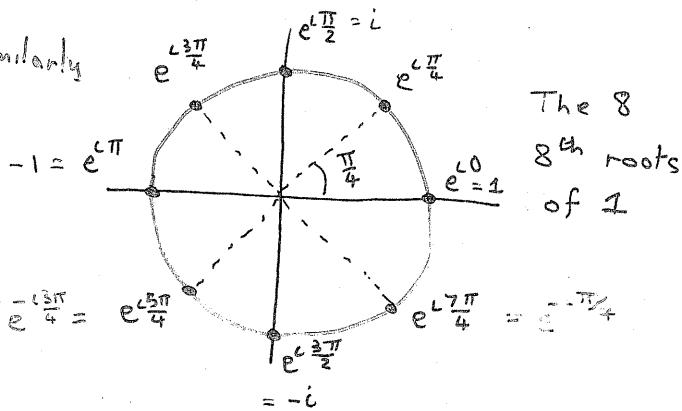
Only the first 3 are different.



$$\text{NB } e^{i\frac{4\pi}{3}} = e^{-i\frac{2\pi}{3}} = (e^{i\frac{2\pi}{3}})^*$$

If z is an n 'th root of 1 so is z^* .

Similarly



The 8
8th roots
of 1

• Log(z)

We can finally do $\log(-1)$

$$> -1 = e^{i\pi} \therefore \log(-1) = i\pi$$

NB if go to non-principal values

$$\text{eq } -1 = e^{i(2n+1)\pi}$$

$$\log(-1) = i(2n+1)\pi \quad n=0, 1, 2 \dots$$

Thus there is no unique defⁿ
we normally take principal value.

> similarly

$$i = e^{i\frac{\pi}{2}} \rightarrow \log(i) = \frac{i\pi}{2}$$

$$> \log(z) = \log(|z|e^{i\theta})$$

$$= \log(|z|) + i\theta$$

is general case.

14.16

• Powers of z

We can work out $z_1 z_2$ for any z_1 & z_2

General case messy - we give some examples:

$$> i^i = (e^{i\frac{\pi}{2}})^i = e^{-\frac{\pi}{2}} = 0.20788 \text{ REAL!}$$

$$> 1^i = e^{i(2m\pi)\frac{1}{\pi}} = e^{i2m} \quad m=0, 1, 2, 3 \dots$$

There are an infinite number of different roots. Case $m=0$ is the Principal Value ($= e^0 = 1$)

$$> 2^i = (e^{\log 2})^i = e^{i\log 2} = \cos(\log 2) + i\sin(\log 2)$$

$$> e^z \text{ either } e^{x+iy} = e^x (\cos y + i\sin y)$$

$$\text{or } z = |z|e^{i\theta} = |z|(\cos\theta + i\sin\theta)$$

$$|z|(\cos\theta + i\sin\theta) = e^{|z|\cos\theta} (\cos(|z|\sin\theta) + i\sin(|z|\sin\theta))$$

14.18

14.6 - Evolution of Numbers

14.1

Leopold Kronecker



Born: 7 Dec 1823 in
Liegnitz, Prussia
(now Legnica, Poland)

Died: 29 Dec 1891 in Berlin,
Germany

"God made the Integers, all the rest
is the work of man"

Kronecker believed that mathematics should deal only with finite numbers and with a finite number of operations. He was the first to doubt the significance of non-constructive existence proofs. It appears that, from the early 1870s, Kronecker was opposed to the use of irrational numbers, upper and lower limits, and the Bolzano-Weierstrass theorem, because of their non-constructive nature. Another consequence of his philosophy of mathematics was that to Kronecker transcendental numbers could not exist.

Progress with numbers

14.20

- counting

1, 2, 3 ...



- zero 0

- Negative integers

-1, -2, -3 ...



- Fractions $\frac{1}{2}, \frac{-17}{65} \dots$

- Irrational $\sqrt{2}, \pi, \sqrt[3]{\pi}$

- Transcendental

π, e etc.

(infinite decimals)

Dense
but gaps

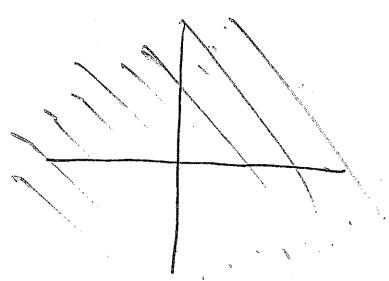
Real Line
is now full

- Complex

- extend real-

Line to

2D PLANE



where
next?

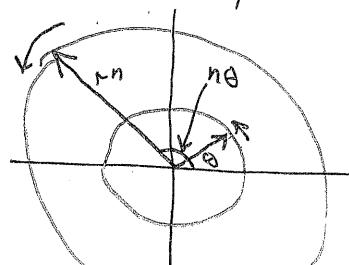
Proof (Not for Exams)

14.22

- we can assume $a_0 \neq 0$
otherwise $z=0$ is a root
I can divide by z to get
polynomial of degree $n-1$
- we can assume $a_n \neq 0$ otherwise
we have a polynomial of degree
 $n-1$. Hence assume $a_n = 1$
- suppose

$$z = r e^{i\theta} \quad (\text{not a root})$$

the locus of z^n as θ goes from
 $0 \rightarrow 2\pi$ is a circle that goes
round the argand diagram n
times



$$z^n = r^n e^{in\theta}$$

In fact we can easily prove the
following theorem:

14.2

"The equation

$$a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0 = 0$$

has n (complex) roots for
all possible (complex) coefficients
 a_0, \dots, a_n .

This is known as "the fundamental
theorem of Algebra"

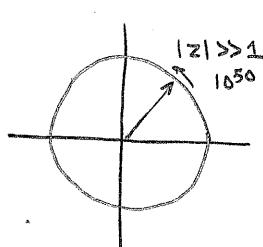
It means that the field of complex
numbers is complete we will
never again encounter something
like $x^2 = -1$ which forced us to
extend our definitions of what
numbers are.

Kronecker - was wrong.

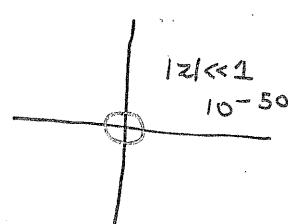
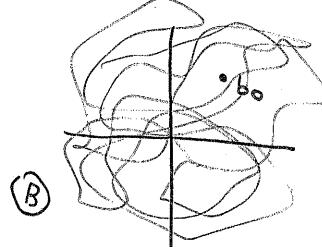
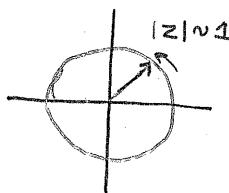
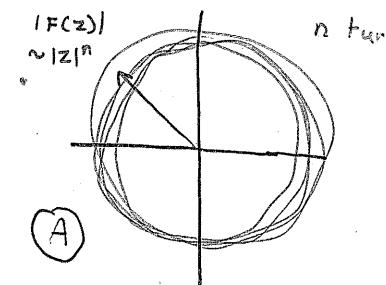
14.22

$$F(z) = z^n + b_{n-1} z^{n-1} + b_{n-2} z^{n-2} + \dots + b_1 z + b_0$$

Locus of z



Locus of $F(z)$

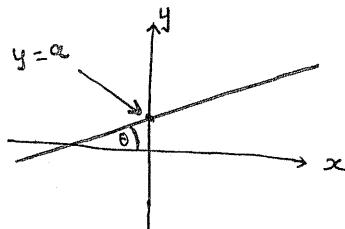


Going $\textcircled{A} \rightarrow \textcircled{B} \rightarrow \textcircled{C}$ locus passes

9 Revision Calculus & Graphs.

- This really is revision
- Agree notation
- Back to 2D world for some time

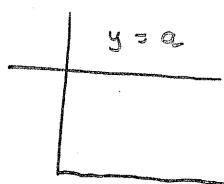
9.1 Straight line



a = [intercept], i.e. value of y at $x = 0$

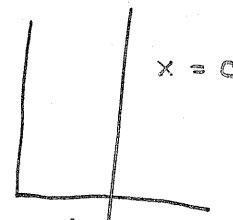
$b = \tan(\theta)$ [slope] of line

(θ is angle line makes with x -axis)



special case

$b = 0$
 $\Rightarrow y$ const.



Special case $\theta = 90^\circ$
 $\tan \theta = \infty$. Line cannot be written $y = a + bx$

The limit $\frac{dy}{dx}$ is the exact value of the slope at point p.

$\frac{\Delta y}{\Delta x}$ is an approximation to the slope which gets better as $\Delta x \rightarrow 0$.

i.e.

$\frac{\Delta y}{\Delta x}$ is the ratio of small quantities

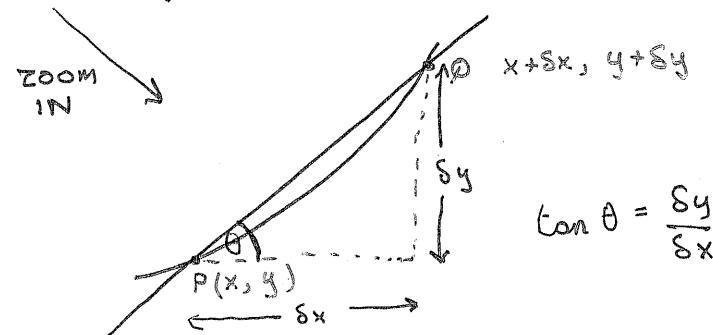
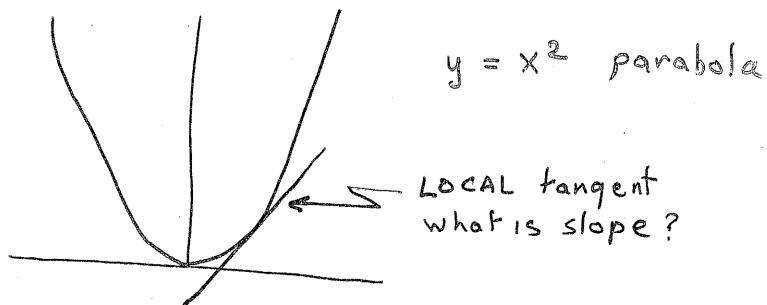
$\frac{dy}{dx}$ is limit which is not the same as $\frac{0}{0}$.

For the time being I will assume you "know" what I mean by "limit" - actually there is a precise def'n, but for now just rely on intuition.

Generalize to

$y = f(x)$ where f is any function of x .

9.2 $\frac{dy}{dx}$ - local slope



$$\Delta y = (x + \Delta x)^2 - x^2 = 2x\Delta x + \Delta x^2$$

$$\therefore \tan \theta = \frac{\Delta y}{\Delta x} = 2x + \Delta x$$

define

$$\frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = 2x$$

This is Calculus.

9.3

The limit $\frac{dy}{dx}$ is the exact value of the slope at point p.

$\frac{\Delta y}{\Delta x}$ is an approximation to the slope which gets better as $\Delta x \rightarrow 0$.

i.e.

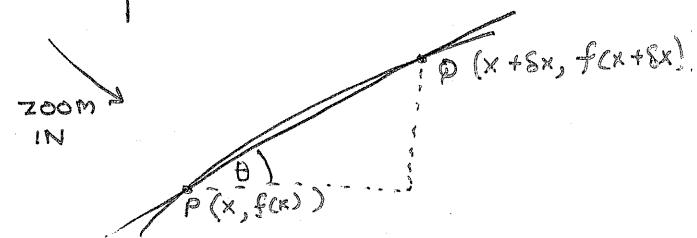
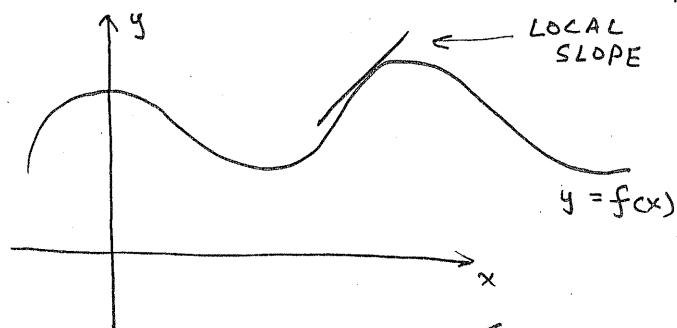
$\frac{\Delta y}{\Delta x}$ is the ratio of small quantities

$\frac{dy}{dx}$ is limit which is not the same as $\frac{0}{0}$.

For the time being I will assume you "know" what I mean by "limit" - actually there is a precise def'n, but for now just rely on intuition.

Generalize to

$y = f(x)$ where f is any function of x .



$$\tan \theta = \frac{f(x + \Delta x) - f(x)}{\Delta x} = \frac{\Delta f}{\Delta x}$$

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta f}{\Delta x} = \frac{df}{dx}$$

exact slope at point p.

Δf just means $f(x + \Delta x) - f(x)$. We have tacitly assumed Δf is small

Tend to use Δf & Δy interchangeably

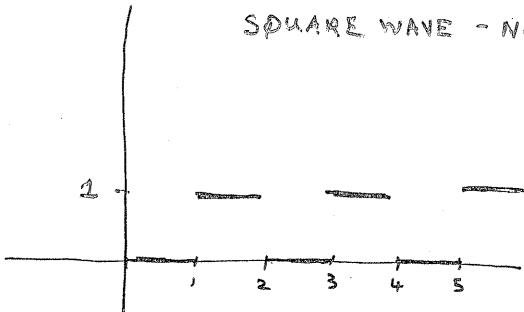
9.3 Existence of $\frac{df}{dx}$

For $\frac{df}{dx}$ to exist at some point x we need:

- $f(x)$ continuous
- $f(x)$ smooth

These have 'natural' definitions - best understood from counter examples

SQUARE WAVE - NOT CONTINUOUS



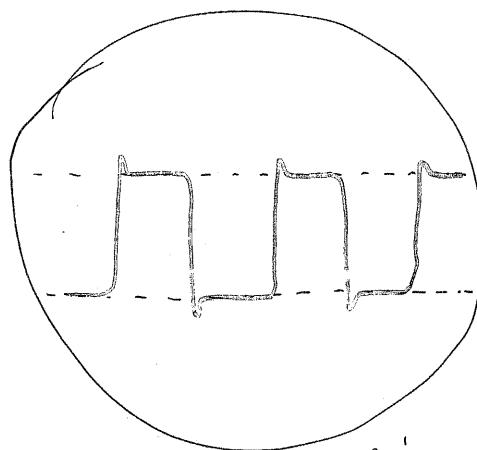
Jumps at $x = 1, 2, 3, 4 \dots$

$$\Delta f = 1 \text{ at } x = 1, 2, 3, 4$$

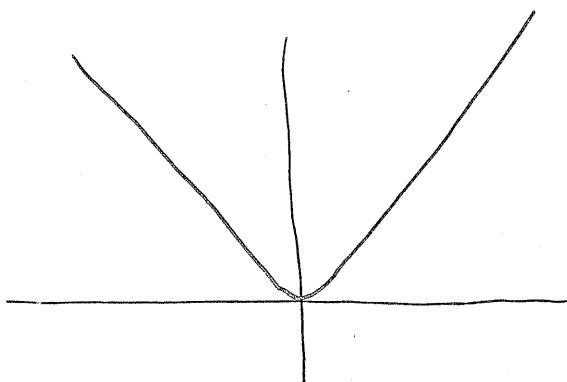
$\therefore \lim_{\Delta x \rightarrow 0} \frac{\Delta f}{\Delta x}$ at $x = 1, 2, 3, 4$ does not exist as Δf finite

$$\text{eg } x = 0.9999 \quad \frac{\Delta f}{\Delta x} = \frac{1}{0.0002} = 5000$$

Real world is well behaved:



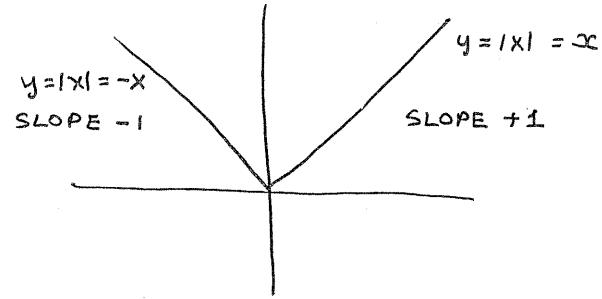
Oscilloscope trace of 'square' wave.



Close up of kink graph.

fn:

$y = |x|$ has kink at $x = 0$



Can't work out $\frac{dy}{dx}$ at $x = 0$

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} \rightarrow \pm 1 \quad \text{sign depends on if } \Delta x > \text{ or } < 0.$$

- Note we could talk about 'left handed' and 'right handed' derivatives - but this not useful for physics.
- The derivative $\frac{dy}{dx}$ is discontinuous at $x = 0$.
- Real Physics tends to be OK, discontinuities & kinks are averaged out.

Aside:

The unreal world of pure maths is NOT always so well behaved.

$$y = f(x)$$

can represent any rule for turning an x -value into a y value.

Eg

$$y = I(x) \text{ where}$$

$$I(x) = 0 \text{ if } x \text{ rational} \\ (\text{eg } \frac{1}{2} \text{ or } \frac{1234}{5678})$$

$$= 1 \text{ if } x \text{ irrational} \\ (\text{eg } \sqrt{2} \text{ or } \pi)$$

This is continuous NOWHERE & differentiable NOWHERE

This function is an interesting counter example to your intuition but has NO ROLE in science (maybe fractals?)

9.4 Special Functions

You MUST know :

$$\bullet \quad y = x^n \quad \frac{dy}{dx} = nx^{n-1}$$

$$\bullet \quad y = e^x \quad \frac{dy}{dx} = e^x$$

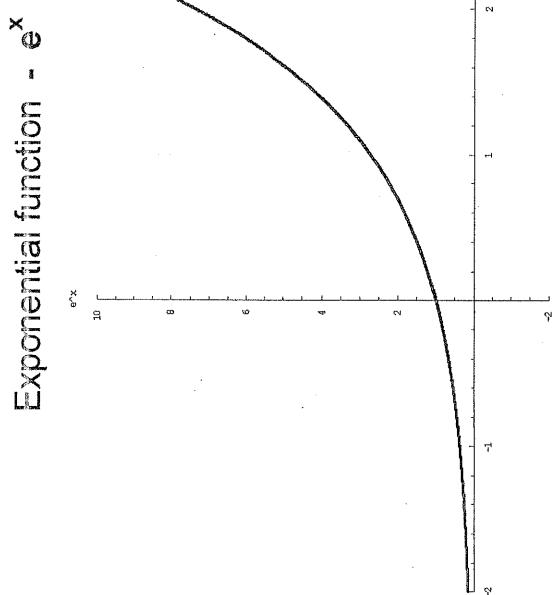
$$\bullet \quad y = \sin x \quad \frac{dy}{dx} = \cos x \quad \text{radians}$$

$$\bullet \quad y = \cos x \quad \frac{dy}{dx} = -\sin x$$

$$\bullet \quad y = \ln x \quad \frac{dy}{dx} = \frac{1}{x}$$

Hopefully you have all seen these. We can build up more complicated things from here.

First a few comments:



Where does e come from?

$$e = 2.718281828\ldots$$

The function e^x is hugely important.

Answer:

Look for function $\exp(x)$ such that

$$\boxed{\begin{aligned}\frac{d}{dx}(\exp(x)) &= \exp(x) \\ \exp(0) &= 1\end{aligned}}$$

⇒ can draw graph

⇒ hence find $\exp(1) = 2.718\ldots = e$

⇒ easy to show

$$\exp(1+x) = \exp(1) \cdot \exp(x)$$

⇒ compare

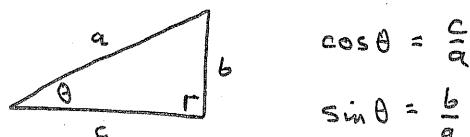
$$e^{1+x} = e^1 \cdot e^x$$

⇒ conclude $\exp(x) \equiv e^x$

9.11

Why do sin & cos keep turning up?

⇒ sin & cos first met in trig



$$\cos \theta = \frac{a}{r}$$

$$\sin \theta = \frac{b}{r}$$

But the Greeks had stumbled on something much more interesting.

$$\Rightarrow \frac{d}{dx}(\sin) = \cos \quad \frac{d}{dx}(\cos) = -\sin$$

$$\sin(0) = 0, \cos(0) = 1$$

⇒ Can draw graphs

$$\Rightarrow \frac{d^2}{dx^2}(\sin) = -\sin$$

$$\frac{d^2}{dx^2}(\cos) = -\cos$$

Solve simple harmonic motion eqn

$$\frac{d^2y}{dx^2} = -y$$

9.5 Rules for complicated expressions

Rule 1 : Product Rule

If $y(x) = f(x)g(x)$

$$\frac{dy}{dx} = \frac{df}{dx}g + f\frac{dg}{dx} = f'g + fg'$$

example $y = x^3 \sin(x)$

$$\frac{dy}{dx} = 3x^2 \sin(x) + x^3 \cos(x)$$

since $f \leftrightarrow x^3$ & $g \leftrightarrow \sin(x)$

Proof:

$$\begin{aligned} y(x+\delta x) &= f(x+\delta x)g(x+\delta x) \\ &= (f(x)+\delta f)(g(x)+\delta g) \\ &= f(x)g(x) + \delta f g(x) + f(x)\delta g + \delta f \delta g \\ \therefore \frac{\delta y}{\delta x} &= \frac{y(x+\delta x)-y(x)}{\delta x} = \frac{\delta f}{\delta x}g(x) + f(x)\frac{\delta g}{\delta x} + \frac{\delta f \delta g}{\delta x} \\ \therefore \lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} &\stackrel{\delta f, \delta g \rightarrow 0}{=} \frac{dy}{dx} = \frac{df}{dx}g + f\frac{dg}{dx} + 0 \end{aligned}$$

Since $\delta f, \delta g$ is second order in small quantities, hence $\frac{\delta f \delta g}{\delta x} \rightarrow 0$ ($\frac{df}{dx} \delta g$ say)

example: $y = \sin(x^3)$

here $f(g) = \sin(g)$

& $g(x) = x^3$

hence $\frac{df}{dg} = \cos(g) = \cos(x^3)$

& $\frac{dg}{dx} = 3x^2$

$$\Rightarrow \frac{dy}{dx} = \frac{df}{dg} \frac{dg}{dx} = \cos(x^3) 3x^2$$

Proof:

$$\begin{aligned} \frac{dy}{dx} &\stackrel{\delta y}{=} \frac{df}{dx} = \lim_{\delta x \rightarrow 0} \frac{\delta f}{\delta x} \\ &= \lim_{\delta x \rightarrow 0} \left(\frac{\delta f}{\delta g} \frac{\delta g}{\delta x} \right) \quad \text{where } \delta g \text{ is change in } g \text{ as } x \rightarrow x + \delta x \\ &= \lim_{\delta g \rightarrow 0} \left(\frac{\delta f}{\delta g} \right) \cdot \lim_{\delta x \rightarrow 0} \left(\frac{\delta g}{\delta x} \right) \\ &= \frac{df}{dg} \frac{dg}{dx} \end{aligned}$$

[Where we have tacitly used a theorem about products of limits]

Rule 1½: Quotient Rule

$$\text{If } y(x) = \frac{f(x)}{g(x)}$$

$$\frac{dy}{dx} = \frac{\frac{df}{dx}g + f\frac{dg}{dx}}{g^2} = \frac{fg' - fg'}{g^2}$$

KNOW

Example:

$$y = \frac{\sin(x)}{x^3}$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{\cos(x)x^3 - \sin(x)3x^2}{x^6} \\ &= \frac{\cos(x)}{x^3} - \frac{3\sin(x)}{x^4} \end{aligned}$$

Proof: special case of:

Rule 2: Chain rule or function of a function

$$y = f(g(x))$$

$$\frac{dy}{dx} = \frac{df}{dg} \frac{dg}{dx}$$

VITAL - make sure you understand this.

Rule 3: x as function of y

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$$

Follows from def'n of $\frac{dy}{dx}$ as ratio.

This one is really simple but very useful if x is a simple f^n of y .

And... that's it! You can now differentiate expressions of any complexity involving the standard functions by REPEATED application of these rules.

You now need lots of practice - I will only do a few examples:

$$(a) y = \frac{1}{1+x^2} \quad \text{this is } (\text{stuff})^{-1}$$

$$\frac{dy}{dx} = -\frac{1}{(1+x^2)^2} \cdot 2x \quad (\text{rule 2})$$

[Or could use rule 1½]

"Limit of product = product of limits"

$$(6) \quad y = \frac{f(x)}{g(x)} = f(x) g^{-1}(x)$$

$$\begin{aligned} \frac{dy}{dx} &= f'(g^{-1}) + f \frac{d}{dx}(g^{-1}) \quad \text{rule 1} \\ &= \frac{f'}{g} + f \cdot -\frac{1}{g^2} g' \quad \text{rule 2} \\ &= \frac{f'}{g} - \frac{fg'}{g^2} = \frac{fg' - fg'}{g^2} \end{aligned}$$

which is the quotient rule.

$$(c) \quad y = \tan x = \frac{\sin(x)}{\cos(x)}$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{\cos x \cdot \cos x + \sin x \sin x}{\cos^2 x} \quad \text{rule } 1\frac{1}{2} \\ &= \frac{1}{\cos^2 x} = \sec^2 x \end{aligned}$$

[NB. $\cos^2 x$ is OK for $(\cos(x))^2$ but
 $\cos^1 x$ is NOT OK for $\frac{1}{\cos x}$]

$$(d) \quad y = \tan^{-1} x$$

This is easy if remember rule 3

(f) Rule 3 also gives:

$$\frac{d}{dx} e^{ax} = ae^{ax}$$

$$\frac{d}{dx} e^{-ax} = -ae^{-ax}$$

$$\frac{d}{dx} e^{-ax^2} = -2ax e^{-ax^2}$$

$$\frac{d}{dx} \sin(ax+b) = a \cos(ax+b)$$

$$\frac{d}{dx} \cos(ax+b) = -a \sin(ax+b)$$

where a & b are constants. These are used all the time - Learn.

If $y = \tan^{-1} x$ then $x = \tan y$

$$\text{hence } \frac{dx}{dy} = \frac{1}{\cos^2 y}$$

$$\text{rule 3: } \frac{dy}{dx} = \cos^2 y = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$$

Worth remembering this one.

$$(e) \quad y^2 \sin y = x$$

Now y is an IMPLICIT f^n of x , we cannot solve to get $y = \text{simple } f^n(x)$

However

$$\frac{dx}{dy} = 2y \sin y + y^2 \cos y$$

$$\text{or } \frac{dy}{dx} = \frac{1}{2y \sin y + y^2 \cos y} \quad \text{rule 3}$$

Final comment - don't get too fixated on x & y . We often want to differentiate w.r.t. other variables, eg time t .

9.18₂

9.6 Higher Derivatives

If $y = y(x)$, then $\frac{dy}{dx}$ is also a f^n of x ,

hence we can differentiate it (provided it is continuous & smooth)

$$\frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d^2 y}{dx^2}$$

Second derivative

Nothing really new - just more calculus

Example $y = x^4 \sin x$

$$\frac{dy}{dx} = 4x^3 \sin x + x^4 \cos x$$

$$\begin{aligned} \frac{d^2 y}{dx^2} &= 12x^2 \sin x + 4x^3 \cos x + 4x^3 \cos x - x^4 \sin x \\ &= (12x^2 - x^4) \sin x + 8x^3 \cos x \end{aligned}$$

again!

$$\begin{aligned} \frac{d^3 y}{dx^3} &= (24x - 4x^3) \sin x + (12x^2 - x^4) \cos x \\ &\quad + 24x^2 \cos x - 8x^3 \sin x \end{aligned}$$

and so on....

$$= (x-1)(x-3)$$

$$y \sim x^2 \text{ as } |x| \rightarrow \infty$$

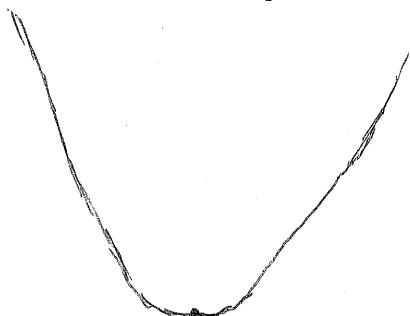
$x = 1$ or 3

$y = 3$ when $x = 0$

expect parabola

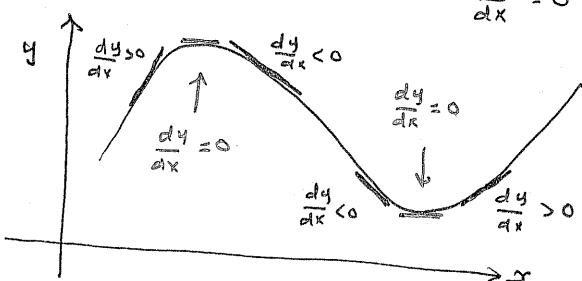
- none

↓ min at $x=2$
by sim
 $y = -1$



10.2 Turning Points

Maxima & minima occur when $\frac{dy}{dx} = 0$



Thus

Maximum $\Leftrightarrow \frac{dy}{dx} = 0 \text{ &} \frac{d^2y}{dx^2} < 0$	LEARN
Minimum $\Leftrightarrow \frac{dy}{dx} = 0 \text{ &} \frac{d^2y}{dx^2} > 0$	

$$\text{check } y = x^2 - 4x + 3$$

$$\frac{dy}{dx} = 2x - 4, \frac{d^2y}{dx^2} = 2 \Rightarrow 2x - 4 = 0 \\ \Rightarrow x = 2 \quad \checkmark$$

$$\frac{d^2y}{dx^2} = 2 \text{ This is } > 0 \therefore \text{minimum } \checkmark$$

N.B. When sketching sometimes $\frac{d^2y}{dx^2}$ is tedious to evaluate at max/min, check using common sense it accessible.

10.2

10.3 Point of Inflexion

What if $\frac{d^2y}{dx^2} = 0$ when $\frac{dy}{dx} = 0$

Actually might have max, min or neither - need to look at plot more carefully.

If $\frac{d^2y}{dx^2} = 0$ we have a point of inflection

(whatever value of $\frac{dy}{dx}$, does not have to $= 0$)

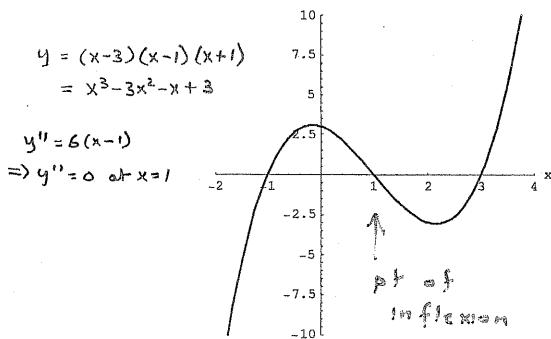
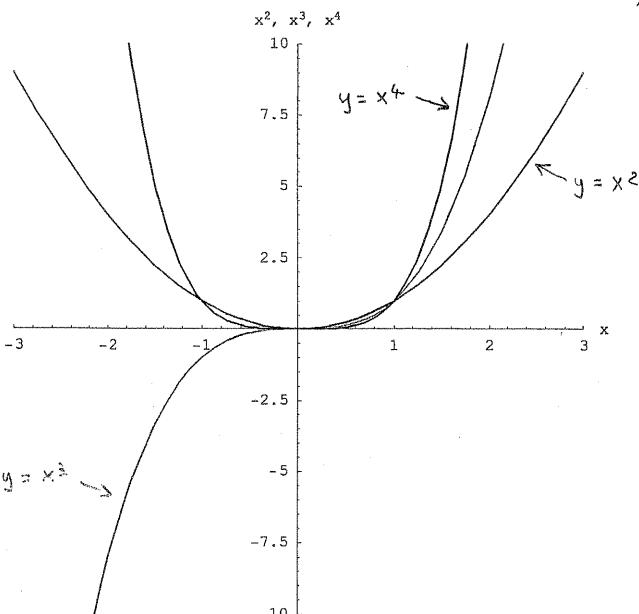
$$\text{eg (1)} y = x^3 \Rightarrow \frac{dy}{dx} = 3x^2 = 0 \text{ at } x=0 \\ \frac{d^2y}{dx^2} = 6x = 0 \text{ at } x=0$$

but neither max nor min

$$(2) y = x^4 \Rightarrow \frac{dy}{dx} = 4x^3 = 0 \text{ at } x=0$$

$$\frac{d^2y}{dx^2} = 12x^2 = 0 \text{ at } x=0$$

This is a min but graph is flatter than parabola $y = x^2$ at $x=0$.



10.4 Example of Cubic.

Sketch $y = x^3 - 3x^2 - x + 3$
 $= (x+1)(x-1)(x-3)$

$\Rightarrow y = 0 \text{ when } x = 3, 1, -1$

$$\frac{dy}{dx} = 3x^2 - 6x - 1$$

\Rightarrow turning points at

$$x = \frac{6 \pm \sqrt{36+12}}{6} = 1 \pm \frac{\sqrt{48}}{3} = 2.155, -0.155$$

$$\frac{d^2y}{dx^2} = 6x - 6$$

$2.155 \leftrightarrow \text{min}$
 $-0.155 \leftrightarrow \text{max}$

$\frac{d^2y}{dx^2} = 0 \text{ when } x=1 \therefore x=1 \text{ point of inflexion.}$

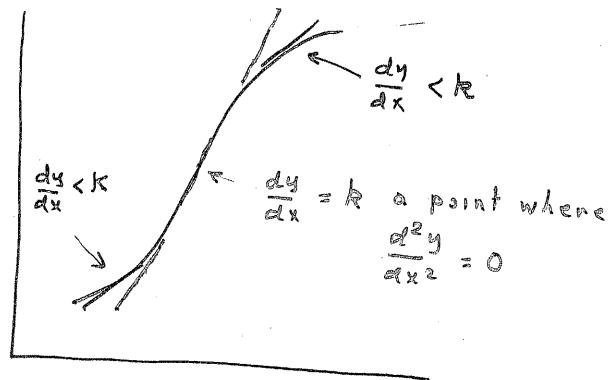
$y \rightarrow +\infty \text{ when } x \rightarrow +\infty$
 $\rightarrow -\infty \text{ " } x \rightarrow -\infty$

Sketch is now easy.

N.B. A cubic cuts x-axis either once or 3 times.

(A quadratic cuts x-axis either never or 2 times (can just touch - counts as 2 roots))

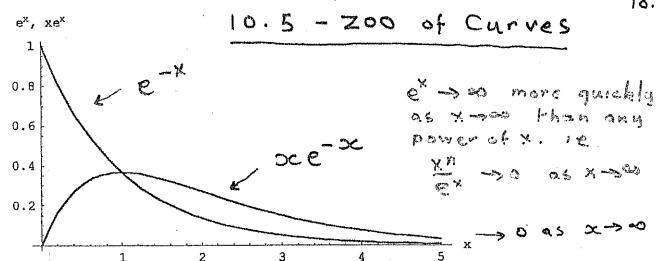
In general a point of inflection occurs when the graph has an 'S' shape



Thus see $\frac{dy}{dx}$ has local max at point of inflection & hence $\frac{d^2y}{dx^2} = 0$!

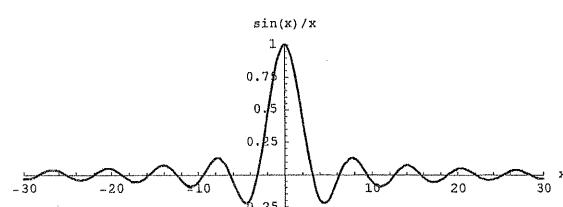
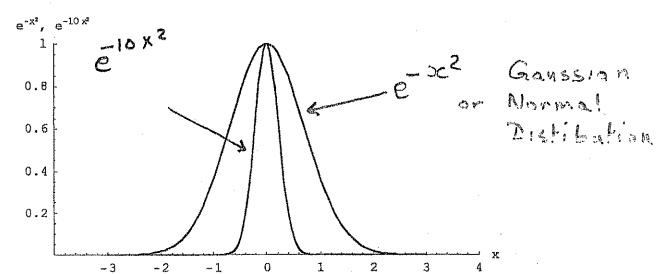
(could be local min as well, I guess sign of $\frac{d^3y}{dx^3}$ will tell you which.)

10.5



10.5 - Zoo of Curves

$e^x \rightarrow \infty$ more quickly as $x \rightarrow \infty$ than any power of x , i.e.
 $\frac{x^n}{e^x} \rightarrow 0$ as $x \rightarrow \infty$



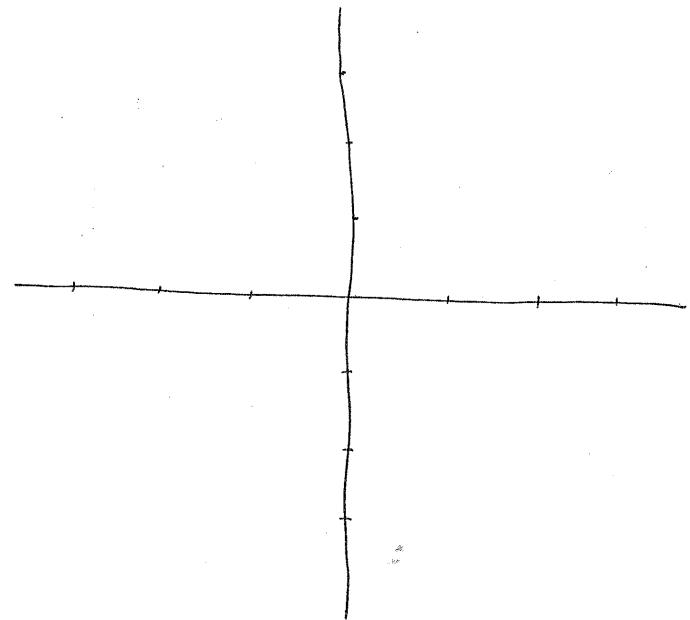
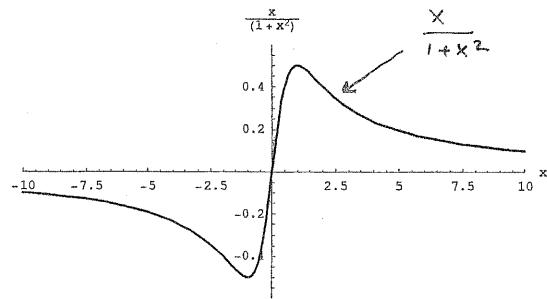
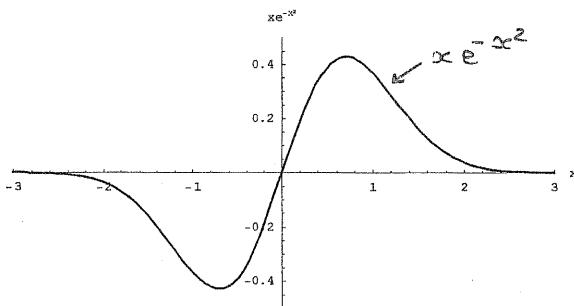
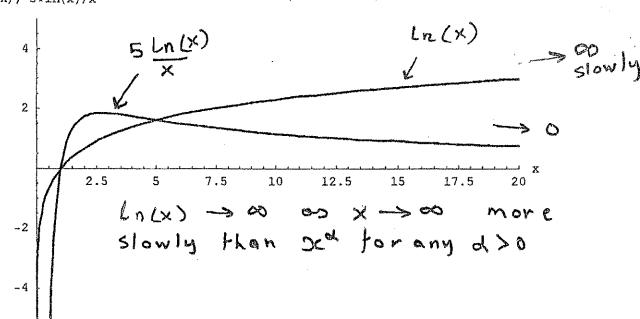
at $x=0$
 $\text{sinc}(x) \equiv$
 $\text{defined as } \frac{\sin x}{x}$

$y = \frac{\sin x}{x} \equiv \text{sinc}(x)$
"diffraction pattern
of slit"

10.6 Example with Singularity

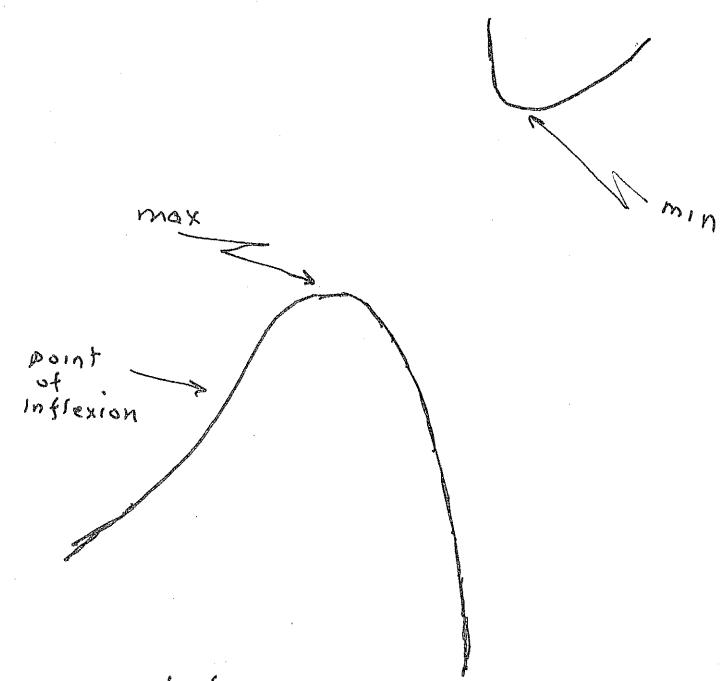
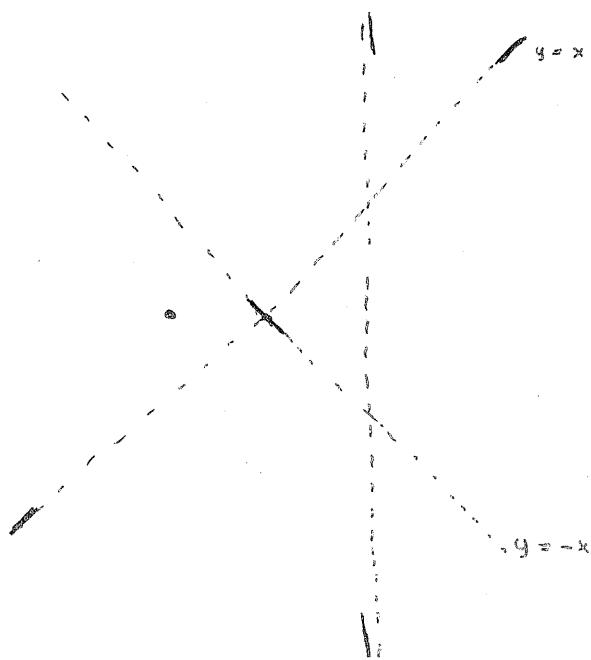
$$y = \frac{x^2+x}{x-1}$$

- as $x \rightarrow \pm\infty$ $y \rightarrow \frac{x^2}{x} = x$ i.e. $y = O(x)$ as $|x| \rightarrow \infty$
- as $x \rightarrow 0$ $y \sim \frac{x}{-1} = -x$
- if $x=1 \pm \epsilon$, ϵ small $y \sim \frac{2}{\epsilon e}$ i.e. $y \rightarrow \infty$ as $x \uparrow 1$
- $y \rightarrow 0$ when $x=0$ or -1



10.8₂

10.8



can use calculus to find max & min etc.

Actually our sketch is NOT quite CORRECT!

$\frac{d^2y}{dx^2}$ does not vanish anywhere \therefore no point of inflection: Our assumption that the curve asymptotes to $y=x$ is not accurate enough to predict shape near origin.

To get better value:

$$y = \frac{x^2+x}{x-1} = \frac{x^2(1+\frac{1}{x})}{x(1-\frac{1}{x})} = x(1+\frac{1}{x})(1-\frac{1}{x})^{-1}$$

$$\& (1-\frac{1}{x})^{-1} = 1 + \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \frac{1}{x^4} + \dots$$

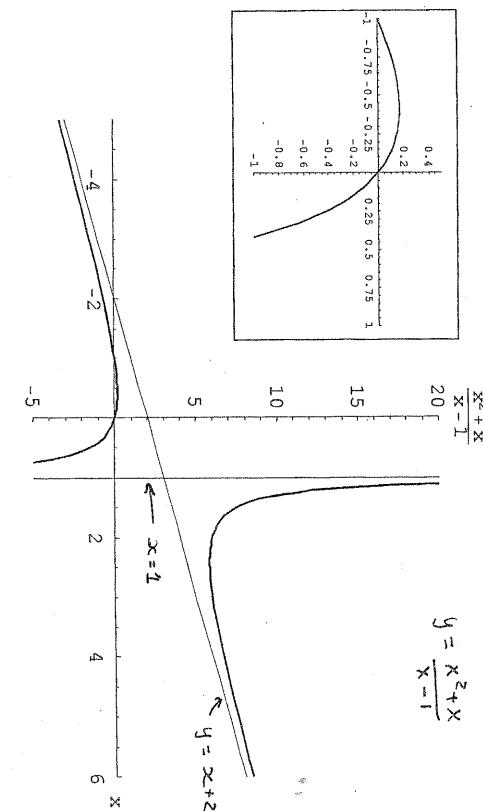
for $|\frac{1}{x}|$ small (< 1)

$$\therefore y = x(1+\frac{1}{x})(1+\frac{1}{x} + \frac{1}{x^2} + \dots)$$

$$= x(1 + \frac{2}{x} + \frac{2}{x^2} + \dots)$$

$$\rightarrow x+2+O(\frac{1}{x}) \quad \text{as } |\frac{1}{x}| \rightarrow 0, |x| \rightarrow \infty$$

Using $y = x+2$ as asymptote gives a good sketch with no need for a point of inflection.



10.7 The Order of Magnitude symbol O

We have used:

$$y \rightarrow x + O(x^2) \quad \text{as } x \rightarrow 0 \quad \text{and so on.}$$

This means $x + \text{term of "Order } x^2\text{"}$

We can give a formal defⁿ to this (but this will not change what you already know).

We say $y = f(x)$ is $O(x^n)$ as $x \rightarrow 0$

$$\text{if } \frac{|f(x)|}{|x^n|} < K \quad \text{for all } |x| < \epsilon$$

where K is a const & ϵ is a (small) constant.

i.e. as $x \rightarrow 0$ $|f(x)|$ is bounded by a fixed multiple of $|x^n|$.

e.g.

$$x^2 + x \text{ is } O(x) \text{ as } x \rightarrow 0$$

$$|x^2 + x| \leq 1.1x \quad \text{for all } |x| < 0.1$$

$x \sin(x)$ is also $O(x)$ as $x \rightarrow 0$

$$\frac{|x \sin(x)|}{|x|} \leq 1 \quad \text{for all } x. \quad \begin{matrix} \text{actually} \\ \text{sin~near~0} \end{matrix}$$

Similarly we say

$$f(x) \text{ is } O(x^n) \text{ as } |x| \rightarrow \infty$$

$$\text{if } \frac{|f(x)|}{|x^n|} < K \quad \text{for all } |x| > A$$

where K is a const & A is another (big) constant

$$x^2 + x \text{ is } O(x^2) \text{ as } x \rightarrow \infty$$

$$\frac{|x^2 + x|}{|x^2|} < 1.1 \quad \text{for all } x > 10$$

And also

$$f(x) \text{ is } O(x^n) \text{ as } x \rightarrow a.$$

$$\text{if } \frac{|f(x)|}{|x^n|} < K \quad \text{for all } |x-a| < \epsilon$$

i.e. as $x \rightarrow a$.

NB $O(x^0) \equiv O(1)$ is allowed & mean the function is bounded by a constant.

e.g. $\sin(x)$ is $O(1)$ for all x .

11. Integration

There are two definitions which will turn out to be the same

11.1 The Indefinite integral

Defined as INVERSE of differentiation, i.e.
if we know $\frac{dy}{dx}$ can we find y ?

we write

$$\text{if } \frac{dy}{dx} = g(x)$$

$$y = \int g(x) dx$$

y is the "integral of g w.r.t. dx ".

Examples:

$$\int x^n dx = \frac{1}{n+1} x^{n+1} + K \quad n \neq -1$$

$$\int x^{-1} dx = \ln|x| + K$$

$$\int \sin \theta d\theta = -\cos \theta + K$$

$$\int \cos \theta d\theta = +\sin \theta + K$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax} + K$$

$$\int e^{-x^2} dx = \text{CAN'T BE DONE}$$

K is an arbitrary constant
since:
 $\frac{dK}{dx} = 0$ can always add K to RHS.

In practice one might try many substitutions before finding one that works.

(C) TRIG SUBSTITUTIONS

Well worth trying to remember these

$$\begin{aligned} & \int \frac{dx}{\sqrt{a^2-x^2}} \quad \text{try } x = a \sin \theta \\ & \quad dx = a \cos \theta d\theta \\ & \rightarrow \int \frac{a \cos \theta d\theta}{a \sqrt{1-\sin^2 \theta}} = \theta + K \\ & \quad = \boxed{\sin^{-1} \frac{x}{a} + K.} \end{aligned}$$

$$\begin{aligned} & \int \frac{dx}{a^2+x^2} \quad \text{try } x = a \tan \theta \\ & \quad dx = \frac{a d\theta}{\cos^2 \theta} = a(1+\tan^2 \theta) d\theta \\ & \rightarrow \int \frac{1}{a} d\theta \\ & \quad = \boxed{\frac{1}{a} \tan^{-1} \frac{x}{a} + K} \end{aligned}$$

$$\begin{aligned} & \int \frac{dx}{\sqrt{a^2+x^2}} = \operatorname{Sinh}^{-1} \frac{x}{a} + K \\ & \quad [\text{Def'n of Sinh later}] \\ & \quad \operatorname{Sinh} x = \frac{e^x - e^{-x}}{2} \\ & \quad \text{actually.} \end{aligned}$$

The above looks promising - unfortunately we cannot find a set of RULES for doing integrals. Instead we have a BOX OF TRICKS sometimes work.

Doing S's requires practice & skill.

11.2 TRICKS for doing integrals

(a) KNOW THE ANSWER

- examples above
- spot exact differential (i.e. g is result of differentiating something)

$$\text{eg } \int x e^{-x^2} dx = -\frac{1}{2} e^{-x^2} + K$$

$$\int \sin x \cos^2 x dx = -\frac{1}{3} \cos^3 x + K.$$

Requires experience to spot these.

(b) SUBSTITUTION

This is the most common trick, effectively formalizes the guesswork in (a)

$$\text{eg } I = \int x e^{-x^2} dx$$

$$\text{try } \boxed{u = x^2} \quad du = 2x dx, \quad dx = \frac{du}{2x}$$

$$\Rightarrow I = \int x e^{-u} \frac{du}{2x} = \frac{1}{2} \int e^{-u} du = -\frac{1}{2} e^{-u} + K \\ = -\frac{1}{2} e^{-x^2} + K.$$

(D) Partial Fractions

Based on manipulations like:

$$\frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{1+x}$$

hence

$$\begin{aligned} \int \frac{dx}{x^2+x} &= \int \frac{dx}{x(x+1)} = \int \frac{dx}{x} - \int \frac{dx}{1+x} \\ &= \ln|x| - \ln|1+x| + K \\ &= \ln\left(\frac{x}{1+x}\right) + K. \end{aligned}$$

(E) Complete square

Aim to use trig substitution, eg:

$$\int \frac{dx}{x^2+2x+10} = \int \frac{dx}{(x+1)^2+9}$$

$$\text{try } u = 1+x, \quad du = dx$$

$$\int \frac{du}{u^2+3^2} = \frac{1}{3} \tan^{-1} \frac{u}{3} + K$$

$$= \frac{1}{3} \tan^{-1} \left(\frac{x+1}{3} \right) + K.$$

N.B. Use 2 consecutive substitutions to do this one

F) INTEGRATION BY PARTS

11.5

NB This is most often used for definite integrals (see later) but can also be used for indefinite integrals.

The method is based on the product rule.

$$\frac{d}{dx}(fg) = \frac{df}{dx}g + f\frac{dg}{dx}$$

$$\therefore f\frac{dg}{dx} = \frac{d}{dx}(fg) - \frac{df}{dx}g$$

$$\therefore \int f\frac{dg}{dx} dx = fg - \int \frac{df}{dx}g dx$$

$$\text{or } \int fg' dx = fg - \int f'g dx$$

useful when

- (a) can integrate g'
- and • (b) $f'g$ is simpler to integrate than fg' .

In words

$$\begin{aligned} \int \text{first. second} &= \text{first. (integral second)} \\ &\quad - \int (\text{derivative first}) \times (\text{integral second}) \end{aligned}$$

TRIG IDENTITIES

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta$$

$$\text{hence } \cos^2 \theta = \frac{1}{2}(1 + \cos 2\theta)$$

$$\& \sin^2 \theta = \frac{1}{2}(1 - \cos 2\theta)$$

$$\Rightarrow \int \cos^2 \theta d\theta = \int \left(\frac{1}{2} + \frac{1}{2} \cos 2\theta\right) d\theta$$

$$= \frac{\theta}{2} + \frac{1}{4} \sin 2\theta + K$$

$$\int \sin^2 \theta d\theta = \int \left(\frac{1}{2} - \frac{1}{2} \cos 2\theta\right) d\theta$$

$$= \frac{\theta}{2} - \frac{1}{4} \sin 2\theta + K$$

And many more examples

Examples of Integration by parts

$$\bullet \int x e^{ax} dx$$

$$= x \left(\frac{1}{a} e^{ax}\right) - \int 1 \left(\frac{1}{a} e^{ax}\right) dx$$

$$= \frac{x}{a} e^{ax} - \frac{1}{a^2} e^{ax}$$

By repeated application we can do things like $\int x^{10} e^{ax} dx$ [But maybe better tricks]

$$\bullet \int x \cos x = x \sin x - \int 1 (\sin x) dx \\ = x \sin x + \cos x$$

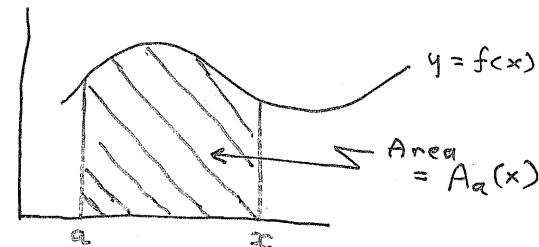
$$\bullet \int \ln x dx = \int (\ln x \cdot 1) dx \\ = \ln x x - \int \frac{1}{x} x dx \\ = x \ln x - x .$$

11.7

11.3 Definite Integral.

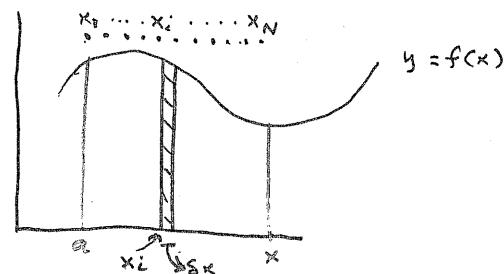
We will show that integrals correspond to areas under graphs.

- Given any (well behaved) $f^n, y = f(x)$ define Area function $A_\alpha(x)$:



A_α is temporary notation, A_α depends on α, x, f

- Think of $A_\alpha(x)$ as limit of sum:



$$A_\alpha(x) = \lim_{N \rightarrow \infty} \sum_{i=1}^N f(x_i) \Delta x$$

$$\Delta x = \frac{x - a}{N}$$

BASIC DEFⁿ.

We see from this "def" that

- if $f(x_i) < 0$ get -ve contributions

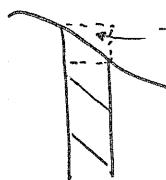
AREA below x-axis is minus

- if $x < a$ sign of Δx reversed

$$A_x(a) = -A_a(x)$$

Sign of AREA changes if go in decreasing x direction

- We have made a small approx (for finite N)

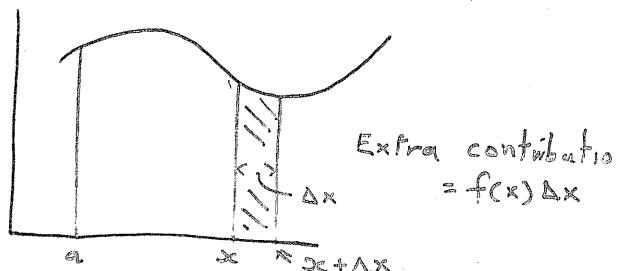


This area included in sum, but is of order $\frac{df}{dx} \Delta x^2$
 \therefore OK in limit

However there is a tacit assumption that f is continuous.

11.8

If Δx is small then $A_a(x)$ & $A_a(x+\Delta x)$ differ only by the addition of one extra slice, w contributing $f(x) \Delta x$



$$\text{if } A_a(x+\Delta x) = A_a(x) + f(x) \Delta x + O(\Delta x^2)$$

thus

$$\lim_{\Delta x \rightarrow 0} \frac{A_a(x+\Delta x) - A_a(x)}{\Delta x} = f(x) + O(\Delta x)$$

$$\frac{d A_a}{d x} = f(x)$$

$$\text{hence } A_a = \int f(x) dx + \text{constant}$$

$$\text{hence } A_a(x) = \int f(x) dx - \int f(x) dx \quad \begin{matrix} \text{at } x=x \\ \text{at } x=\infty \end{matrix}$$

$$\text{Since } A_a(a) = 0$$

In particular for some fixed $x = b$

11.10

$$A_a(b) = \int_{a \leftarrow x=b}^{f(x) dx} - \int_{x=a}^{f(x) dx}$$

$$\equiv \int_a^b f(x) dx$$

This is the familiar definite integral, notice it is a number equal to area under graph.

To paraphrase:

The area under the graph $y = f(x)$ in the range $a \leq x \leq b$ ($x \in [a, b]$) can be calculated by integrating $f(x)$ to get, say, $F(x)$. Then

$$\text{Area} = F(b) - F(a)$$

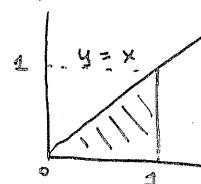
$$\text{where } \frac{dF}{dx} = f(x)$$

KNOW

any choice of integration const K will do for $F(x)$ because K cancels in the subtraction

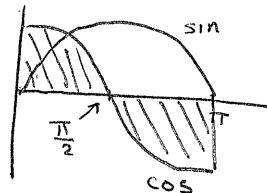
11.11 Examples

$$\bullet f(x) = x, x \in [0, 1]$$



$$\int_0^1 x dx = \left[\frac{1}{2} x^2 \right]_0^1 = \frac{1}{2} 1^2 - \frac{1}{2} 0^2 = \frac{1}{2}$$

$$\bullet f(x) = \cos(x), x \in [0, \pi]$$

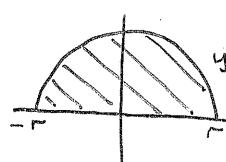


$$\int_0^\pi \cos x dx = [\sin x]_0^\pi = 0 - 0 = 0 \checkmark$$

Remember sign convention for areas.

$$\text{Similarly } \int_0^\pi \sin x dx = [-\cos x]_0^\pi = -(-1) - -(1) = 2$$

$$\bullet \text{Area under semicircle, radius } r.$$



$$y = \sqrt{r^2 - x^2}$$

$$\text{Area} = \int_{-r}^r \sqrt{r^2 - x^2} dx$$

$$\text{subst } x = r \sin \theta \quad dx = r \cos \theta d\theta$$

$$\Rightarrow \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{r^2 - r^2 \sin^2 \theta} r \cos \theta d\theta$$

$$\begin{aligned}
 &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} r^2 \cos^2 \theta \, d\theta \\
 &= r^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2} (1 + \cos 2\theta) \, d\theta \\
 &= r^2 \left[\frac{\theta}{2} + \frac{1}{4} \sin 2\theta \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\
 &= r^2 \left[\frac{\pi}{4} + \frac{1}{4} \sin \pi - \left(-\frac{\pi}{4} \right) - \frac{1}{4} \sin(-\pi) \right] \\
 &= \frac{1}{2} \pi r^2 \quad \checkmark
 \end{aligned}$$

- Even & Odd functions.

If $f(x) = -f(-x)$	this is ODD
$f(x) = f(-x)$	this is EVEN

LEARN

$$\int_{-a}^a f(x) \, dx = 0 \quad \text{if } f \text{ is odd}$$

$$\int_{-a}^a f(x) \, dx = 2 \int_0^a f(x) \, dx \quad \text{if } f \text{ even}$$

$$\sin(\theta) \text{ is an odd fn} \quad \therefore \int_{-\alpha}^{\alpha} \sin \theta \, d\theta = 0$$

$$\cos(\theta) \text{ is an even fn}$$

$$\therefore \int_{-\alpha}^{\alpha} \cos \theta \, d\theta = 2 \int_0^{\alpha} \cos \theta \, d\theta$$

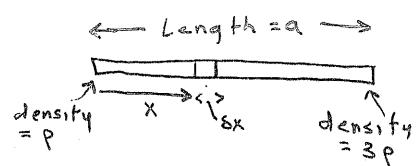
11.5 Integral as sum

$$\text{Recall } \int_a^b f(x) \, dx = \lim_{\delta x \rightarrow 0} \sum_{x_i=a}^b f(x_i) \delta x$$

This is very useful, the sum on RHS occurs all the time - it does not have to be AREA.

Example 1:

"Rod of length a has density that varies smoothly from ρ to 3ρ . Find mass."



$$\rho \text{ at distance } x \text{ from end} = \rho(1 + \frac{2x}{a})$$

$$\therefore \text{mass of element } \delta x = \rho(1 + \frac{2x}{a}) \delta x$$

$$\therefore \text{Total mass} = \sum_{x=0}^a \rho(1 + \frac{2x}{a}) \delta x$$

$$\rightarrow \int_0^a \rho(1 + \frac{2x}{a}) \, dx$$

$$= \left[x + \frac{x^2}{a} \right]_0^a \rho = (a + \frac{a^2}{a}) \rho = [2\rho a]$$

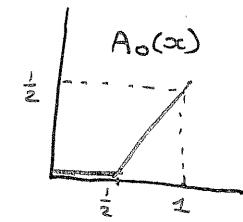
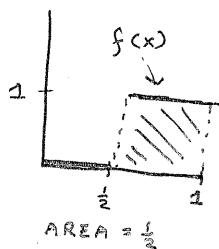
- We might have guessed this, 2ρ is average density.
- If ρ varies in more complicated way we have to do the \int .

11.4 Example with $f(x)$ NOT-Continuous

$$\text{What if } f(x) = 0 \quad x < \frac{1}{2}$$

$$f(x) = 1 \quad x > \frac{1}{2}$$

What is area under graph for $0 \leq x \leq 1$?



Value of $A_o(x)$ is AREA under graph.

Graph of $A_o(x)$ has KINK at $x = \frac{1}{2}$

$$\therefore \frac{dA_o(x)}{dx} \neq f(x) \text{ at } x = \frac{1}{2}$$

since LHS does not exist - however we try to define $f(\frac{1}{2})$.

NAIVE attempt to use recipie fails

$$\text{Area} = \int_0^1 f(x) \, dx, \quad f = 1 \text{ at top limit } \int_1^x = x \\ f = 0 \text{ at bottom limit } \int_0^0 = 0 \\ \therefore \text{area} = 1 - 0 = 1 \quad \times$$

CORRECT METHOD - split range.

$$\text{Area} = \int_0^{\frac{1}{2}} f(x) \, dx + \int_{\frac{1}{2}}^1 f(x) \, dx = \int_0^{\frac{1}{2}} 0 \, dx + \int_{\frac{1}{2}}^1 1 \, dx \\ = [0]_{0}^{\frac{1}{2}} + [x]_{\frac{1}{2}}^1 = 0 - 0 + 1 - \frac{1}{2} = \frac{1}{2} \quad \checkmark$$

11.6 Example 2 Stirlings Formula for $n!$

notice:

$$\ln(n!) = \ln 1 + \ln 2 + \ln 3 + \dots + \ln(n)$$

$$= \sum_{x_i=1}^n \ln(x_i) 1$$

$$\approx \int_1^n \ln(x) \, dx \quad \text{for large } n$$

$$= [x \ln x - x]_1^n$$

$$= n \ln n - n + 1$$

Ignore

$$\ln(n!) = n \ln n - n \quad \dots A$$

$$\text{hence } e^{\ln(n!)} = e^{n \ln n - n} = e^{n \ln n} e^{-n}$$

$$\therefore n! = n^n e^{-n} \quad \dots B$$

In fact A is a good approx to $\ln n!$ since fractional error $\rightarrow 0$ as $n \rightarrow \text{big}$.

B is a BAD approx since a small error in $\ln n!$ leads to a factor in $n!$ A better formula is

$$(n!) = n^n e^{-n} \sqrt{2\pi n} \quad \dots C$$

122013682599111006870123878542304692625357434280319284219
 2413588385845373153881997605496447502203281863013616477
 1482035841633787220781772004807852051593292854779075719
 3933060377296085908627042917454788242491272634430567017
 327076946106280231045264421887878946575477149863494367
 7810376442740338273653974713864778784954384895955375379
 9042324106127132698432774571554630997720278101456108118
 8373709531016356324432987029563896628911658974769572087
 9269288712817800702651745077684107196243903943225364226
 0523494585012991857150124870696156814162535905669342381
 3008856249246891564126775654481886506593847951775360894
 0057452389403357984763639449053130623237490664450488246
 6507594673586207463792518420045936969298102226397195259
 7190945217823331756934581508552332820762820023402626907
 8983424517120062077146409794561161276291459512372299133
 4016955236385094288559201872743379517301458635757082835
 5780158735432768888680120399882384702151467605445407663
 5359841744304801289383138968816394874696588175045069263
 6533817505547812864000
 000
 00

n	$n!$	$\frac{n^n}{n!} e^{-n} \sqrt{2\pi n}$	% error
1	1.	0.367879	0.922137
5	120.	21.0561	1.65069
10	3.6788×10^6	453999	0.828996
20	2.4329×10^{18}	2.16128×10^{17}	0.415765
50	3.04141×10^{64}	1.71307×10^{63}	0.166526
100	9.33262×10^{187}	3.72008×10^{186}	0.0832883
200	$7.886578674 \times 10^{374}$	$2.223835978 \times 10^{373}$	0.041658
500	$1.220136826 \times 10^{134}$	$2.176512754 \times 10^{132}$	0.016653
1000	$4.023872601 \times 10^{2567}$	$5.0759590 \times 10^{2565}$	0.00833299

Comparison of Stirlings Formulae for $n!$
 (using Mathematica)

11.6 Additional Properties of Definite Integrals

- $I = \int_a^b f(x) dx = - \int_b^a f(x) dx$
 Swap limits \leftrightarrow get $-$ sign
- I is a f^n of a & $b \therefore$ can differentiate w.r.t. either a or b
- $\frac{\partial I}{\partial b} = f(b) \quad$ from defⁿ
- $\frac{\partial I}{\partial a} = -f(a) \quad$ (by swapping limits)

[Note notation for partial derivatives, means diff w.r.t. indicated variable & keep all others constant. You will see a LOT more of $\frac{\partial f}{\partial x}$ etc]

- $\int_a^b f(x) dx = \int_a^b f(t) dt = \int_a^b f(p) dp$
 x is "dummy variable" of integration
 It does not matter what symbol we choose

11.16

Suppose $f = f(x, \lambda)$ i.e. f depends on some parameter λ (e.g. $e^{-\lambda x}$)

Suppose also $\frac{\partial f}{\partial \lambda}$ exists.

$$\left[\frac{\partial}{\partial \lambda} (e^{-\lambda x}) = -x e^{-\lambda x} \quad \text{for example} \right]$$

Then

$$I = \int_a^b f(x, \lambda) dx \quad \text{is a } f^n \text{ of } \lambda \\ (\text{but not } \infty !)$$

From defⁿ of I as limit of sum \dots :

$$\begin{aligned} \frac{d}{d\lambda} \int_a^b f(x, \lambda) dx &= \frac{d}{d\lambda} \sum f(x, \lambda) \Delta x \\ &= \sum \frac{\partial f}{\partial \lambda} \Delta x \\ &= \int_a^b \frac{\partial f}{\partial \lambda} dx \end{aligned}$$

- Can swap order of differentiation and integration w.r.t. a parameter.

Important example:

$$I = \int_0^\infty e^{-ax} dx = \left[-\frac{1}{a} e^{-ax} \right]_0^\infty = \frac{1}{a}$$

I is a f^n of the parameter a as expected.

$$\frac{dI}{da} = \int_0^\infty \frac{\partial}{\partial a} (e^{-ax}) dx = \frac{\partial}{\partial a} \left(\frac{1}{a} \right)$$

$$\therefore \int_0^\infty -x e^{-ax} dx = -\frac{1}{a^2}$$

$$\text{or } \boxed{\int_0^\infty x e^{-ax} dx = \frac{1}{a^2}}$$

Do it again

$$\int_0^\infty x^2 e^{-ax} dx = \frac{1 \cdot 2}{a^3}$$

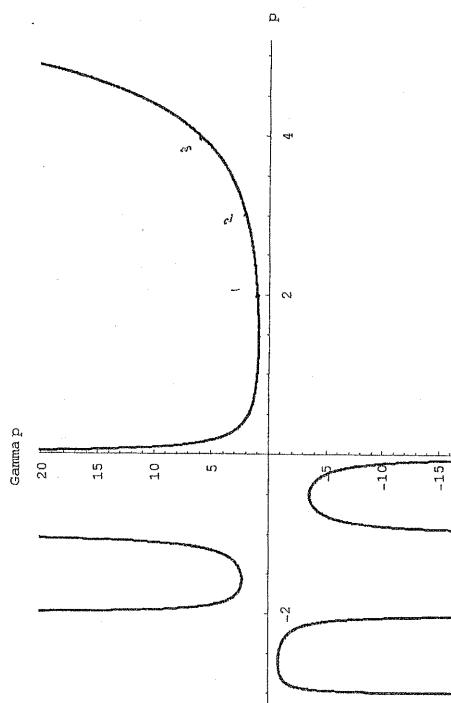
$$\int_0^\infty x^3 e^{-ax} dx = \frac{1 \cdot 2 \cdot 3}{a^4}$$

:

$$\boxed{\int_0^\infty x^n e^{-ax} dx = \frac{n!}{a^{n+1}}}$$

This is much quicker than integration by parts, but only works for some limits.

Plot of Gamma Function in Range -3 to 5



Finally note that if we put $a=1$ get

$$n! = \int_0^\infty x^n e^{-x} dx$$

Another formula for $n!$ but this time exact.

In fact in more advanced work the Gamma Function

$$\Gamma(p) = \int_0^\infty x^{p-1} e^{-x} dx \quad p > 0$$

is defined as a generalization of $n!$ to real numbers

$$\Gamma(p) = (p-1)! \quad p=1, 2, 3, \dots$$

(You may meet Γ 's in part 2B)

11.7 Schwartz Inequality

Hints at generalization of dot product

Triangle inequality for 3D vectors:

$$\begin{aligned} \text{Diagram: Two vectors } \underline{a} \text{ and } \underline{b} \text{ originating from the same point, forming an angle } \theta. \\ |\underline{a} + \underline{b}|^2 &\leq (|\underline{a}| + |\underline{b}|)^2 \\ &\Rightarrow |\underline{a}|^2 + 2\underline{a} \cdot \underline{b} + |\underline{b}|^2 \\ &\leq |\underline{a}|^2 + 2|\underline{a}||\underline{b}| + |\underline{b}|^2 \end{aligned}$$

$$\text{or } \underline{a} \cdot \underline{b} \leq |\underline{a}| |\underline{b}|$$

[Alternatively just note $\cos \theta \leq 1$]

Get equality if \underline{a} & \underline{b} parallel.

write $(\underline{a} \cdot \underline{b})^2 \leq |\underline{a}|^2 |\underline{b}|^2$ in component

$$(a_x b_x + a_y b_y + a_z b_z)^2$$

$$\leq (a_x^2 + a_y^2 + a_z^2)(b_x^2 + b_y^2 + b_z^2)$$

Think of extending these sums to get integrals; maybe:

$$\left(\int_a^b f g dx \right)^2 \leq \int_a^b f^2 dx \int_a^b g^2 dx$$

Schwartz's Inequality - easy to prove

- L. Twin brothers L.

Proof

$$\int_a^b (f + \lambda g)^2 dx \geq 0 \text{ for any } f(x), g(x) \& \lambda$$

$$\int f^2 + 2\lambda \int fg + \lambda^2 \int g^2 \geq 0 \quad [\text{Leave out limits & } dx \text{ to save writing}]$$

can assume $\int f^2 \neq 0$ otherwise $f=0$ everywhere
 & $\boxed{\quad}$ is true, both sides zero.

$$\therefore 1 + 2\lambda \frac{\int fg}{\int f^2} + \lambda^2 \frac{\int g^2}{\int f^2} \geq 0$$

$$\left(1 + \lambda \frac{\int fg}{\int f^2}\right)^2 - \lambda^2 \left(\frac{\int fg}{\int f^2}\right)^2 + \lambda^2 \frac{\int g^2}{\int f^2} \geq 0$$

choose $\lambda = -\frac{\int f^2}{\int fg}$ so that $(\quad)^2 = 0$

(if $\int fg = 0$ $\boxed{\quad}$ must be true, LHS = 0)

$$-\lambda^2 \left(\frac{\int fg}{\int f^2}\right)^2 + \lambda^2 \frac{\int g^2}{\int f^2} \geq 0$$

$$\Rightarrow (\int fg)^2 \leq \frac{(\int f^2)^2}{\int f^2} \int g^2 = \int f^2 \int g^2$$

QED.

Comments

- $\int_a^b f(x) g(x) dx$ is called the "Inner Product" of f & g . It is a generalization of the dot product to the vector space of real functions.

- From our def" of \int as limit of sum expect

$$\left(\sum_{i=1}^N a_i b_i\right)^2 \leq \left(\sum_{i=1}^N a_i^2\right) \left(\sum_{i=1}^N b_i^2\right)$$

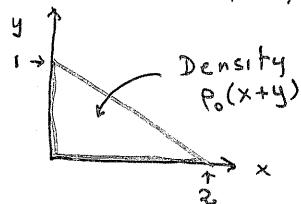
Another Schwarz inequality but with vectors having N components instead of 3.

12 MULTIPLE INTEGRALS

12.1

Extend the idea of doing SUMMATIONS by means of definite integrals to functions of more than one variable.

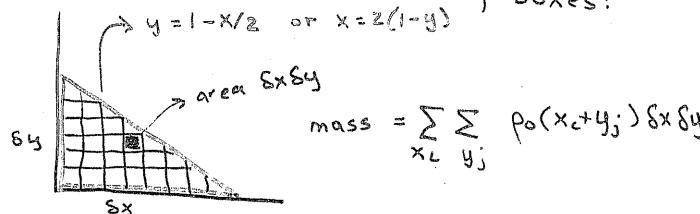
12.1 Double Integrals in Cartesians



Triangular sheet density = $p_0(x+y)$

What is total mass?

sum contributions from many boxes:



To do this double sum we can

(a) either: let $\delta x \rightarrow 0$ get $\int dx$, then let $\delta y \rightarrow 0$ get $\int dy$.(b) or: let $\delta y \rightarrow 0$ get $\int dy$, then let $\delta x \rightarrow 0$ get $\int dx$.Method (a) $\delta x \rightarrow 0$ first

$$\begin{aligned} \text{mass} &= \sum_{y_j} \left(\sum_{x_i} p_0(x_i + y_j) \delta x \right) \delta y \\ &\rightarrow \sum_{y_j} \delta y \int_{x=0}^{2(1-y_j)} p_0(x+y_j) dx \end{aligned}$$

Top Limit
Depends on
Local value of
 y_j

$$\text{and } \sum_{y_j} \delta y \int_{x=0}^{2(1-y_j)} p_0(x+y) dx$$

MUST DO x integral first:

$$\int_{y=0}^1 dy p_0 \left[\frac{x^2}{2} + xy \right]_0^{2(1-y)} = 2(1-y)$$

$$= \int_{y=0}^1 2(1-y) p_0 dy$$

$$= p_0 \left[2y - \frac{2}{3}y^2 \right]_0^1 = \boxed{p_0}$$

*Footnote

$$\begin{aligned} \left[\frac{x^2}{2} + xy \right]_0^{2(1-y)} &= \frac{4(1-y)^2}{2} + 2(1-y)y \\ &= 2(1-y)(1-y+4) \\ &= 2(1-y) \end{aligned}$$

Method (b) $\delta y \rightarrow 0$ first

$$\text{mass} = \sum_{x_i} \left(\sum_{y_j} \rho_0(x_i + y_j) \delta y \right) \delta x$$

$$\xrightarrow{\delta y \rightarrow 0} \sum_{x_i} \delta x \int_0^{1-x_i/2} \rho_0(x_i + y) dy$$

and then

$$\xrightarrow{\delta x \rightarrow 0} \int_0^2 dx \int_0^{1-x/2} \rho_0(x+y) dy$$

MUST DO y integral first:

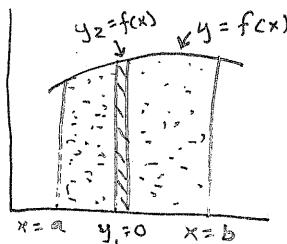
$$\begin{aligned} & \int_0^2 dx \rho_0 \left[xy + \frac{y^2}{2} \right]_0^{1-x/2} \\ &= \frac{1}{2} x - \frac{3}{8} x^2 + \frac{1}{2} \\ &= \int_{x=0}^2 \rho_0 \left(\frac{1}{2} x - \frac{3}{8} x^2 + \frac{1}{2} \right) dx \\ &= \rho_0 \left[\frac{x^2}{4} - \frac{x^3}{8} + \frac{x}{2} \right]_0^2 = \boxed{\rho_0} \end{aligned}$$

Same answer as expected.

Footnote

$$\begin{aligned} \left[xy + \frac{y^2}{2} \right]_0^{1-x/2} &= \left(1 - \frac{x}{2} \right) x + \left(1 - \frac{x}{2} \right)^2 \frac{1}{2} \\ &= \left(1 - \frac{x}{2} \right) \left(x + \frac{1}{2} - \frac{x}{4} \right) = \left(1 - \frac{x}{2} \right) \left(\frac{3x}{4} + \frac{1}{2} \right) \\ &= \frac{3x}{4} + \frac{1}{2} - \frac{3x^2}{8} - \frac{x}{4} = \frac{x}{2} - \frac{3x^2}{8} + \frac{1}{2} \end{aligned}$$

We can now see that the definite integral we did in 11.3 is just a special case of the area integral:



$$\iint dxdy = \int_a^b dx \int_{y_1=0}^{y_2=f(x)} dy = \int_a^b f(x) dx$$

Thus in the standard definite integral we have already done the y integration to get area of vertical slice.

12.2 Other slice geometries

In evaluating

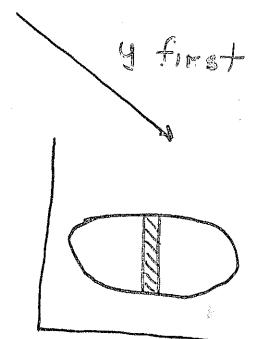
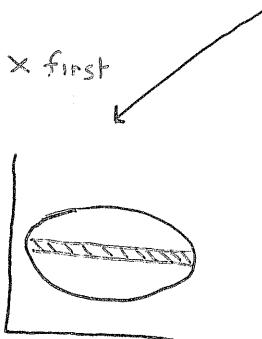
$$\iint f(x,y) dA \quad dA = \text{area element}$$

region

we can try other geometries.

Recap, for any $f^n f(x,y)$ the double integral

$\iint f(x,y) dx dy$ can do in 2 ways:



For some fixed y
get x-slice from:

$$\int_{x_1(y)}^{x_2(y)} f(x,y) dx$$

Then do y integral

For some fixed x
get y-slice from:

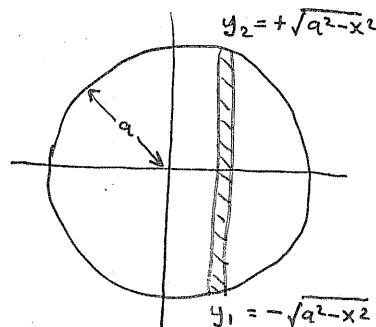
$$\int_{y_1(x)}^{y_2(x)} f(x,y) dy$$

Then do x integral

N.B. if $f(x,y) = 1$ we simply get area of region

i.e. $\iint dxdy = \text{area}$

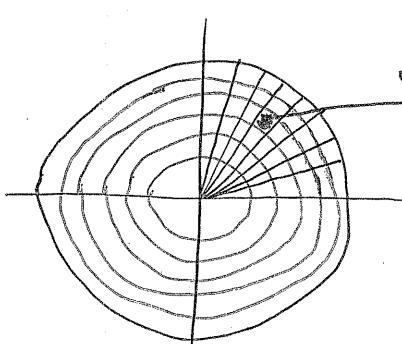
Area of circle revisited:



$$\begin{aligned} A &= \iint dxdy \\ &\text{circle} \\ &= \int_{x=-a}^a dx \int_{y_1}^{y_2} dy \\ &= \int_{x=-a}^a dx \left[y \right]_{y_1}^{y_2} \end{aligned}$$

$$\therefore A = 2 \int_{-a}^a \sqrt{a^2 - x^2} dx = \pi a^2 \text{ as before}$$

OR USE PLANE POLAR COORDINATES!



Area element
 $\delta A = r \delta r \delta \theta$

CONSTANT r contour
CONSTANT theta contour

Thus in plane polar coords

$$\text{Area} = \iint r dr d\phi \quad (\text{limit of sum})$$

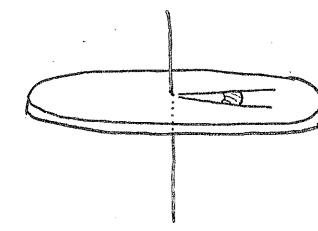
circle
r=a
 $\phi=2\pi$

$$= \int r dr \int d\phi = \frac{\alpha^2}{2} 2\pi = \pi a^2$$

NB Limits for r
& ϕ do not depend
on each other - big
simplification.

Note the area elements $r dr d\phi$ are wedge shaped not rectangles - but this is OK they still fill the circle.

In general if a 2D problem has circular sym it is best to use plane polars



density ρ
= mass/unit area
Assume ρ constant.

$$\begin{aligned} \text{defn } I &= \sum (\text{elements of mass}) \times (\text{distance from axis})^2 \\ &= \sum \delta m r^2 \\ &= \sum r^2 \underbrace{\rho r dr d\phi}_{\text{area element}} \\ &\rightarrow \int_{r=0}^{r=a} \int_{\phi=0}^{\phi=2\pi} r^3 \rho dr d\phi \\ &= \left[\frac{r^4}{4} \right]_0^a 2\pi \rho = \frac{\pi}{2} a^4 \rho \end{aligned}$$

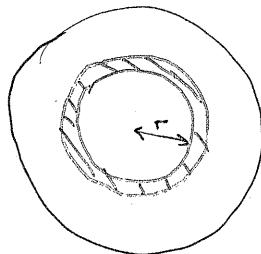
But $\pi a^2 \rho = \text{total mass of disk, say } m$

$$\therefore \boxed{I = \frac{1}{2} m a^2}$$

(which is correct)

L12

You may have seen this calculation done by summing circular slices



green area
 $\approx 2\pi r \delta r$

Slice of thickness δr at radius r has area $2\pi r \delta r$, \therefore mass of slice

$$= \rho 2\pi r \delta r$$

\therefore contrib to moment of inertia I

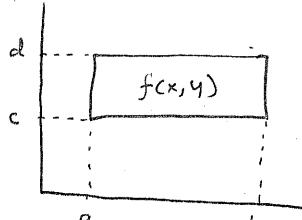
$$SI = \rho 2\pi r \delta r \times r^2$$

$$\therefore I = \int_0^a 2\pi \rho r^3 dr \rightarrow \frac{1}{2} m a^2 \text{ as before}$$

Effectively we have done the ϕ integration in our heads (using sym, $\int d\phi = 2\pi$) And ended up with just a 1D \int . This approach is FINE.

Possible Simplification

Note if the region of integration has a simple shape the limits might not depend on each other:



$$I = \iint_{\text{region}} f(x, y) dx dy = \int_a^b \int_c^d f(x, y) dx dy$$

If ALSO $f(x, y) = g(x) h(y)$ i.e. $f(x, y)$ is a product of a f^n of x & a f^m of y , eg $f(x, y) = y^2 e^x$ rather than $\sqrt{x^2 + y^2}$

$$I = \int_{y=c}^d \int_{x=a}^b g(x) h(y) dx dy$$

$$= \left(\int_{y=c}^d h(y) dy \right) \left(\int_{x=a}^b g(x) dx \right)$$

Product of 2 1-dim integrals. A similar "Separation of Variables" is also possible for polar coords & in 2D.

2.3 Volume Integrals

12.10

It is straight forward to extend these ideas to 3D volume integrals.

For general f^n $f(x, y, z) \equiv f(\xi)$

$$\text{Volume Integral} = \lim_{\delta V \rightarrow 0} \sum \underset{\substack{\text{position} \\ \text{of volume} \\ \text{element}}}{f(\xi_i)} \underset{\substack{\text{volume} \\ \text{element}}}{\delta V}$$

$$\rightarrow \iiint f(\xi) dV$$

$$\text{In Cartesians } dV = dx dy dz$$

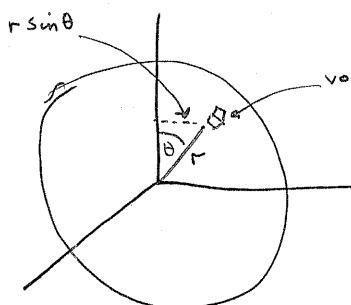
$$\text{In Cylindrical polars } dV = r dr d\theta dz$$

$$\text{In Spherical polars } dV = r^2 \sin\theta dr d\theta d\phi$$

MUST KNOW

Choose your coordinate system wisely - i.e. exploit symmetries.

Example 2 Moment of Inertia of Sphere



volume element $r^2 \sin\theta dr d\theta d\phi$
uniform density ρ
= mass/unit volume

$$I = \sum \delta m (r \sin\theta)^2 = \sum \rho r^2 \sin^2\theta \delta V$$

$$\rightarrow \iiint_{r=0}^a \iiint_{\theta=0}^{\pi} \iiint_{\phi=0}^{2\pi} \rho r^4 \sin^3\theta dr d\theta d\phi$$

$$= \rho \int_0^a r^4 dr \int_0^{\pi} \sin^3\theta d\theta \int_0^{2\pi} d\phi$$

$$\rho \frac{a^5}{5} \quad \downarrow \frac{4}{3} \quad \downarrow 2\pi$$

$$\frac{8\pi\rho a^5}{15} = \boxed{\frac{2}{5} M a^2} \quad \text{correct}$$

$$M = \frac{4}{3}\pi a^3 \rho$$

Example 1 Volume of Sphere

If $f(r) = 1$, $\iiint dV = \text{Volume}$

use spherical polars, $0 \leq r \leq a$,

$0 \leq \theta \leq \pi$

$0 \leq \phi \leq 2\pi$

$$\text{Volume} = \int_{r=0}^a \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} r^2 \sin\theta dr d\theta d\phi$$

$$= \int_{r=0}^a r^2 dr \int_{\theta=0}^{\pi} \sin\theta d\theta \int_{\phi=0}^{2\pi} d\phi$$

$$\left[\frac{r^3}{3} \right]_0^a \quad \left[-\cos\theta \right]_0^{\pi} \quad \left[\phi \right]_0^{2\pi}$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$\boxed{\frac{4}{3}\pi a^3}$$

N.B. Limits all independent :- very easy.

Footnote

$$\int_0^{\pi} \sin^3\theta d\theta = \frac{4}{3} ?$$

substitute $C = \cos\theta$

then $dc = -\sin\theta d\theta$

$C = -1$ when $\theta = \pi$

$C = +1$ when $\theta = 0$

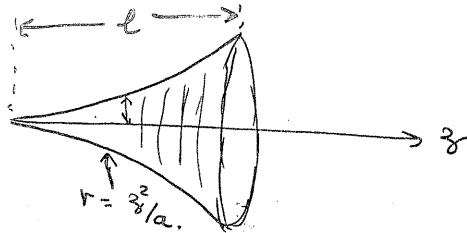
$$\begin{aligned} &\Rightarrow \int_{+1}^{-1} -\sin^2\theta dc \\ &= \int_{+1}^{-1} \sin^2\theta dc \\ &= \int_{+1}^{-1} (1-C^2) dc \\ &= \left[C - \frac{C^3}{3} \right]_{+1}^{-1} \\ &= \frac{4}{3} \end{aligned}$$

Note the substitution $C = \cos\theta$ is very often helpful in spherical polar integrals.

Think of $\sin\theta d\theta$ as $-d(\cos\theta)$ & see

Example 3 Volume of Parabolic Cone

12.14



Use cylindrical polar coordinates

$$z : 0 \leftrightarrow l \quad \phi : 0 \leftrightarrow 2\pi$$

$$r : 0 \leftrightarrow \frac{z^2}{a} \quad \text{depends on } z$$

$$\begin{aligned} \text{Volume} &= \iiint_{\text{cone}} dV = \iiint_{\text{cone}} r dr d\phi dz \\ &= \int_{z=0}^l \int_{r=0}^{z^2/a} \int_{\phi=0}^{2\pi} r dr d\phi dz \\ &= 2\pi \int_0^l dz \int_0^{z^2/a} r dr \\ &= 2\pi \int_0^l dz \left[\frac{r^2}{2} \right]_0^{z^2/a} \\ &= 2\pi \int_0^l \frac{z^4}{2a^2} dz = 2\pi \left[\frac{z^5}{10a^2} \right]_0^l \\ &= \frac{\pi l^5}{5a^2} \end{aligned}$$

do # 1

$$I^2 = 2\pi \int_0^\infty r e^{-r^2} dr$$

$$\begin{aligned} \text{put } u &= r^2 \\ du &= 2r dr \end{aligned}$$

$$\begin{aligned} I^2 &= \pi \int_0^\infty e^{-u} du \\ &= \pi \left[-e^{-u} \right]_0^\infty = \pi \\ \therefore I &= \sqrt{\pi} \quad (\text{Answer must be } > 0 \therefore \text{NOT } -\sqrt{\pi}) \end{aligned}$$

$$\begin{aligned} \text{i.e. } \int_{-\infty}^{\infty} e^{-x^2} dx &= \sqrt{\pi} \\ \text{B. } \int_0^{\infty} e^{-x^2} dx &= \frac{1}{2}\sqrt{\pi} \quad (\text{even f''}) \end{aligned}$$

TRY TO LEARN

In STATISTICS we are interested in $\int_{-\infty}^{\infty} e^{-x^2/2\sigma^2} dx$ - This is now simple
put $u^2 = \frac{x^2}{2\sigma^2}$
 $u = \frac{x}{\sqrt{2\sigma}} \quad du = \frac{dx}{\sqrt{2\sigma}}$

12.4 Gaussian Integrals

We CAN evaluate $\int_{-\infty}^{\infty} e^{-x^2} dx$ by means of an elegant trick. [We can't do this \int with finite limits].

$$\text{Let } I = \int_{-\infty}^{\infty} e^{-x^2} dx$$

Then $I = \int_{-\infty}^{\infty} e^{-y^2} dy$ since x & y are dummy variables

$$\begin{aligned} \text{Hence } I^2 &= \int_{-\infty}^{\infty} e^{-y^2} dy \int_{-\infty}^{\infty} e^{-x^2} dx \\ &= \int_{y=-\infty}^{\infty} \int_{x=-\infty}^{\infty} e^{-y^2} e^{-x^2} dx dy \\ &= \iint_{\text{2D Plane}} e^{-(x^2+y^2)} dx dy \end{aligned}$$

CHANGE TO POLAR COORDS

$$dx dy \leftrightarrow r dr d\phi \quad (\text{Area element})$$

NB shapes differ!

$$x^2 + y^2 \leftrightarrow r^2$$

$$I^2 = \int_{r=0}^{\infty} \int_{\phi=0}^{2\pi} e^{-r^2} r dr d\phi$$

AN EASY INTEGRAL

12.16

get

$$N(\epsilon) = \int_{-\infty}^{\infty} e^{-x^2/2\epsilon^2} dx$$

$$= \sqrt{2\epsilon} \int_{-\infty}^{\infty} e^{-u^2} du = \sqrt{2\epsilon} \sqrt{\pi}$$

Hence

$$N(x, \epsilon) = \sqrt{\frac{1}{2\pi\epsilon^2}} e^{-x^2/2\epsilon^2}$$

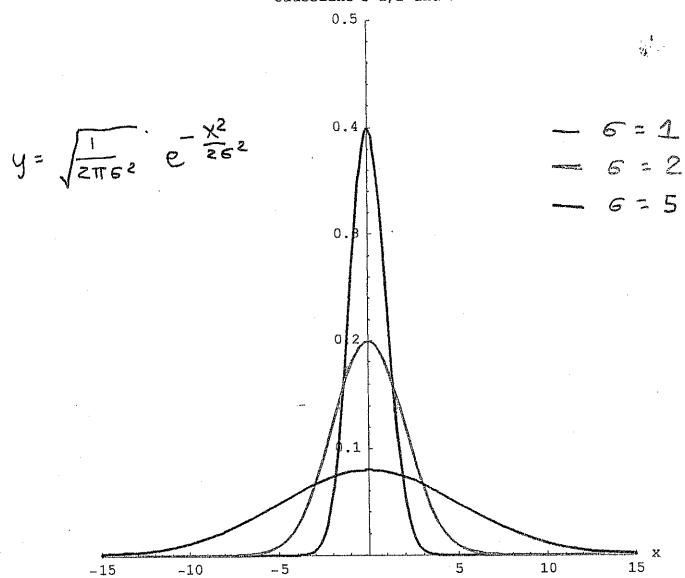
satisfies $\int N(x, \epsilon) dx = 1$.

This is the Normal Distribution f' for mean $\mu=0$ and standard deviation σ . It is VERY IMPORTANT!

For mean $\mu \neq 0$ we have

$$N(\mu, \epsilon) = \sqrt{\frac{1}{2\pi\epsilon^2}} e^{-\frac{(x-\mu)^2}{2\epsilon^2}}$$

Gaussian Distributions

Gaussians $\sigma=1, 2$ and 5

- N.B.
- $-\sigma \leq x \leq \sigma$ has 68.33% of area
 - $-\sigma \leq x \leq 2\sigma$ has 95.45% "
 - $-\sigma \leq x \leq 3\sigma$ has 99.73% "

These numbers v. important in Statistics

- > The fact that the above is wrong is a pity because it is a very plausible attempt to generalize the correct method of substitution in 1D Integrals.

$$\text{eg } x = a \sin \theta$$

$$dx = a \cos \theta d\theta.$$

- > We have run out of time for this topic. You will meet many variations

$$\text{eg surface integrals } \int_{\text{surface}} f(x) dS$$

$$\text{line integrals } \int F \cdot d\ell$$

$$\text{vector surface integrals } \int V \cdot dS$$

etc ...

Just remember each one represents the sum of many small contributions any you should be able to make sense of the notation...

12.5 A FINAL COMMENT

Notice I said $dxdy \leftrightarrow r dr d\phi$ when we changed from Cartesian to Polar Coords. This is because each is the correct definition for AREA elements in the respective coord systems.

I did NOT say $dxdy = r dr d\phi$

Students sometimes make the following MISTAKE $x = r \cos \phi, y = r \sin \phi$

$$\therefore dx = dr \cos \phi - r \sin \phi d\phi$$

$$\& dy = dr \sin \phi + r \cos \phi d\phi$$

$$\therefore dxdy = (dr \cos \phi - r \sin \phi d\phi)(dr \sin \phi + r \cos \phi d\phi)$$

$$= \sin \phi \cos \phi dr^2 + (\cos^2 \phi - \sin^2 \phi) r dr d\phi$$

$$- \sin \phi \cos \phi r^2 d\phi^2$$

The RHS is a MESS & leads to WRONG answers. The RHS still represents the rectangle $dxdy$ in cartesians it is NOT suitable for an area element in polars.

> There is a correct way of doing this which is too advanced for this course. It INVOLVES JACOBIANS.

13 - Power Series

3.1 Geometric Progression

Consider

$$S_N = 1 + x + x^2 + \dots + x^N \equiv \sum_{n=0}^{n=N} x^n$$

Easy to show

$$S_N = \frac{1 - x^{N+1}}{1 - x} \quad \text{--- (*)}$$

Proof by Induction

$$\text{for } N=1 \quad \frac{1-x^2}{1-x} = \frac{(1+x)(1-x)}{1-x} = 1+x$$

so (*) is true for $N=1$ (& $N=0$)

Assume true up to $N=M$ say

$$\begin{aligned} \text{Then } S_{M+1} &= S_M + x^{M+1} \\ &= \frac{1 - x^{M+1}}{1 - x} + x^{M+1} = \frac{1 - x^{M+1} + x^{M+1} - x}{1 - x} \\ &= \frac{1 - x^{M+2}}{1 - x} \quad \therefore (*) \text{ also true for } N=M+1 \end{aligned}$$

Hence, by Induction, (*) True for all N .

Now if $|x| < 1$ we know $x^n \rightarrow 0$ as $N \rightarrow \infty$

13.2

> Thus for example, with $x = \frac{1}{2}$

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots = \frac{1}{1-\frac{1}{2}} = 2.$$

for fixed x , with $|x| < 1$

$$S_N \rightarrow \frac{1}{1-x} \text{ as } N \rightarrow \infty$$

i.e. The INFINITE SERIES

$$\lim_{N \rightarrow \infty} \sum_{n=0}^N x^n \rightarrow \frac{1}{1-x} \text{ as } N \rightarrow \infty$$

for $|x| < 1$

We say the series CONVERGES to this limit (i.e. sum gets closer & closer as we add more terms).

On the other hand

$$\text{if } x \geq 1 \quad S_N \rightarrow \infty \text{ as } N \rightarrow \infty$$

if $x = -1$ S_N oscillates between 0 and 1 as $N \rightarrow \infty$

& if $x < -1$ S_N oscillates between increasing + & - values as $N \rightarrow \infty$

In all these cases we say the series Diverges.

This allows up to explain Xeno's Paradox (~ 450 BC)

> The series $\sum_{n=0}^{\infty} x^n$ is actually quite important, it can be thought of as either the sum of a GP or an example of the Binomial expansion, i.e.

$$\frac{1}{1-x} = (1-x)^{-1} = 1 + x + x^2 + x^3 + \dots$$

for $|x| < 1$.

> We now move on to general power series.

Zeno's Paradox

Zeno of Elea (southern Italy)

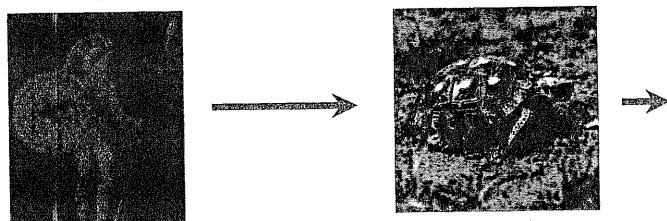
Born: about 490 BC

Died: about 425 BC



13.4

Achilles can NEVER catch up with the tortoise because it will always have moved forward when Achilles gets to where it was.



13.42

$$\text{Time to go } d = \frac{d}{10V} \quad \text{tort goes } \frac{d}{10} \text{ in this time}$$

$$\text{“ “ “ } \frac{d}{10} = \frac{d}{100V} \quad \text{“ “ “ } \frac{d}{100} \text{ “ }$$

$$\text{“ “ “ } \frac{d}{100} = \frac{d}{1000V} \quad \text{“ “ “ } \frac{d}{1000} \text{ “ }$$

Total Time is FINITE &

$$= \frac{d}{10V} \left(1 + \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} \right) = \frac{d}{10V} \cdot \frac{1}{1 - 1/10}$$

$$= \frac{d}{9V} \quad \therefore \text{Reach Tort at dist } \frac{10}{9} d$$

In general if a series is absolutely convergent this is good - anything reasonable you do will work.

If a series is convergent but not absolutely convergent there are dangers.

Example

The Harmonic Series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} \quad \text{Diverges}$$

But

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \quad \text{Converges}$$

However if you try to change the order you sum the terms you can get different answers

Eg take all the -ve terms first
get $-\frac{1}{2}(1 + \frac{1}{2} + \frac{1}{3} + \dots)$ divergent!

3.2 General Power Series - Convergence

3.5

$$\text{Let } P(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots$$

$P(x)$ represents a function of x defined for the range of x values for which the series converges (ie $\sum_0^N a_n x^n \rightarrow \text{limit}$).

$P(0) = a_0 \therefore$ OK for $x=0$, what about other x values?

Some Statements - without proof (see books or wait till easier)

These apply to a general series

$$\sum_{n=0}^{\infty} u_n \quad (\text{same as power series for } x=1)$$

- if $\sum_0^{\infty} |u_n|$ is convergent the series is said to be ABSOLUTELY CONVERGENT and $\sum_0^{\infty} u_n$ is CONVERGENT. We can rearrange terms in any order

3.7 The RATIO TEST

This is the GOLD STANDARD for seeing if a power series converges.

The series

$\sum_0^{\infty} u_n$ is Absolutely Convergent

$$\text{if } \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| \rightarrow K < 1$$

i.e. the ratio of successive terms tends to a limit between 0 & 1.

$$\text{if } \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| \rightarrow K > 1$$

The series diverges.

$$\text{if } \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| \rightarrow 1$$

we need another test.

[The RATIO Test is actually very easy to prove, just compare each term to a geometric progression with ratio R $R < K < 1$]

3.8 Apply RATIO Test to

$$P(x) = \sum_{n=0}^{\infty} a_n x^n$$

Absolutely Convergent if

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1} x^{n+1}}{a_n x^n} \right| \\ = \left| \frac{a_{n+1}}{a_n} \right| |x| \rightarrow K < 1. \end{aligned}$$

Thus if

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| \rightarrow A$$

The series is Absolutely Convergent

$$\text{for } |x| < \frac{1}{A}$$

& divergent for $|x| > \frac{1}{A}$

$|x| = \frac{1}{A}$ defines a "circle of convergence" inside of which the series converges, outside of which the series diverges.

13.3 Taylor Series

3.9

By choosing different values for the a_n 's we can get lots of different power series. But can we find a set of a_n 's to give a specific function, e.g. $\sin(x)$?

Suppose

$$f(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots$$

& the series converges to a function we can differentiate.

$$f(0) = a_0 \text{ by putting } x=0$$

$$\frac{df}{dx} \equiv f' = a_1 + 2a_2 x + 3a_3 x^2 + \dots$$

$$\therefore \frac{df}{dx} \Big|_{x=0} \equiv f'_0 = a_1 \text{ by putting } x=0$$

$$\frac{d^2f}{dx^2} \equiv f'' = 2a_2 + 3.2a_3 x + 4.3.2a_4 x^2 + \dots$$

$$\frac{d^2f}{dx^2} \Big|_{x=0} \equiv f''_0 = 2a_2 \text{ from } x=0$$

$$\text{or } a_2 = \frac{1}{2} f''_0$$

Strictly the above series is a MacLaurin series (but people don't use this name much). It is a special case of

$$f(k+h) = f(k) + h f'_k + \frac{h^2}{2!} f''_k + \dots + \frac{h^n}{n!} f^n_k + \dots$$

$$\text{where } f^n_k \equiv \frac{df}{dx^n} \Big|_{x=k}$$

The expansion of f about some point $x=k$ rather than $x=0$.

- We often sum just a few terms to get an approximation to a function. Taylors Theorem allows us to estimate the error:

$$f(k+h) = f(k) + h f'_k + \dots + \frac{h^{n-1}}{(n-1)!} f^{n-1}_k \leftarrow \text{if stop here}$$

$$+ \frac{h^n}{n!} f^n_{x=k+\alpha} \leftarrow \text{get this error}$$

for some $\alpha : 0 \leq \alpha \leq h$.

Keep going :

$$f''' = 3.2.1 a_3 + 4.3.2 a_4 x + 5.4.3 a_5 x^2 + \dots$$

$$\rightarrow a_3 = \frac{1}{3!} f'''|_{x=0}$$

And so on to get

$$a_n = \frac{1}{n!} f^n_0 \quad (f^n_0 \equiv \frac{df}{dx^n} \Big|_{x=0})$$

$$f(x) = f(0) + x f'_0 + \frac{x^2}{2!} f''_0 + \frac{x^3}{3!} f'''_0 + \dots + \frac{x^n}{n!} f^n_0 + \dots$$

This is correct provided all the series are absolutely convergent, in particular, The Taylor Series converges if

$$\lim_{n \rightarrow \infty} \left\{ \frac{x^{n+1} f^{n+1}_0}{(n+1)! f^n_0 x^n} \right\} < 1$$

$$\lim_{n \rightarrow \infty} \left\{ \frac{x}{n+1} \frac{f^{n+1}_0}{f^n_0} \right\} < 1$$

The $\frac{1}{n+1}$ is very helpful!

13.4 Examples

We can now write down power series for all our old friends.

- e^x all the derivatives are e^x & are equal to 1 at $x=0$, hence

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots + \frac{x^n}{n!} + \dots$$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \dots (-1)^n \frac{x^n}{n!} + \dots$$

MUST LEARN

Apply ratio test

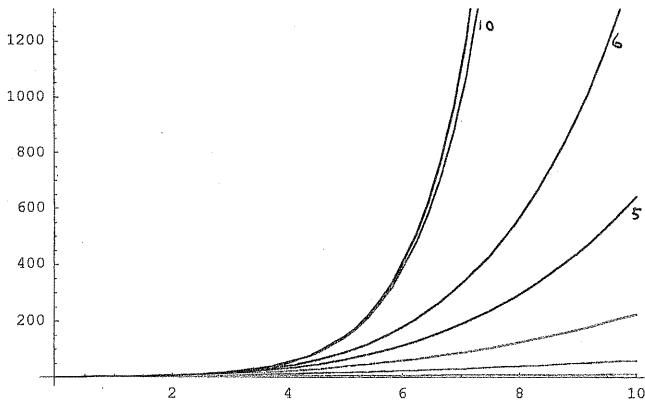
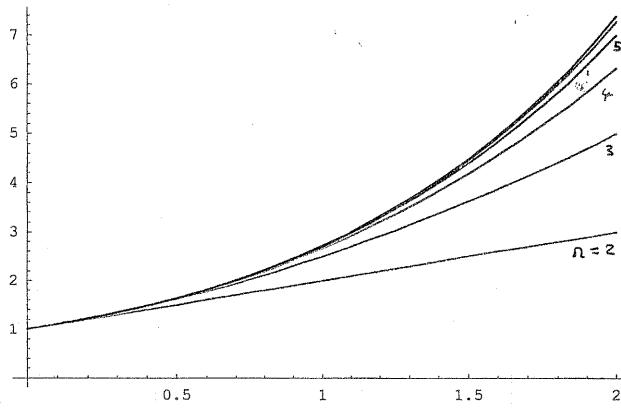
$$\left| \frac{x^{n+1} n!}{(n+1)! x^n} \right| = \left| \frac{x}{n+1} \right| \rightarrow 0 \text{ as } n \rightarrow \infty$$

for any fixed x

Absolutely Convergent for All values of x .

The series converges rapidly for any fixed x once $n > \infty$.

Comparison of $y = e^x$ and sum of Taylor series up to n terms



Sin x & Cos x

$$\text{since } \frac{d}{dx}(\sin x) = \cos(x) \quad \cos(0) = 1$$

$$\& \frac{d}{dx}(\cos x) = -\sin(x) \quad \sin(0) = 0$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

MUST KNOW

$\sin x$ is an ODD f^n and has odd powers
 $\cos x$ is an EVEN f^n and has even powers

Both series are absolutely convergent
 for all x . [Proof left to you].

Useful Approximations for $|x| \ll 1$

$$e^x = 1 + x + O(x^2)$$

$$\sin x = x + O(x^3)$$

$$\cos x = 1 - \frac{x^2}{2} + O(x^4)$$

Remember

Numerical Comparison of e^x and sum of Taylor series up to n terms

Example $x = 1$

N	Sum	Error	% Error
1	1	1.71828	63.2121
2	2.	0.718282	26.4241
3	2.5	0.218282	8.03014
4	2.66667	0.0516152	1.89882
5	2.70833	0.0099485	0.365985
10	2.71828	3.02886×10^{-7}	0.0000111425

3.15

Binomial Series

$$\text{take } f(x) = (1+x)^p$$

$$\text{then } f'(x) = p(1+x)^{p-1}, \quad f'' = p(p-1)(1+x)^{p-2} \dots$$

Hence

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!} x^2 + \dots + \frac{p(p-1)\dots(p-n+1)}{n!} x^n + \dots$$

SERIES TERMINATES for $n=0, 1, 2, \dots$ otherwise LEARN infinite.

Apply Ratio Test:

$$R = \left| \frac{(p-n)x^{n+1}}{(n+1)!} \frac{n!}{x^n} \right| = \left| \frac{p-n}{n+1} x \right|$$

$\rightarrow |x| \text{ as } n \rightarrow \infty$

∴ Abs convergent $|x| < 1$

Divergent $|x| > 1$

($|x| = 1$ can't say in general
 need other tests)

13.16

Examples of Binomial Expansion:

13.14

one more to see pattern:

13.18

$$(1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x + \frac{\frac{1}{2} \cdot -\frac{1}{2}}{2!} \frac{x^2}{2!} + \frac{\frac{1}{2} \cdot -\frac{1}{2} \cdot -\frac{3}{2}}{3!} \frac{x^3}{3!}$$

$$= 1 + \frac{1}{2}x - \frac{x^2}{8} + \frac{x^3}{16} \dots$$

$$(1+x)^{-1} = 1 - x + \frac{1 \cdot 2}{2!} x^2 - \frac{1 \cdot 2 \cdot 3}{3!} x^3$$

$$= 1 - x + x^2 - x^3 \dots$$

$$\text{Sim } (1-x)^{-1} = 1 + x + x^2 + x^3 + \dots \quad (\text{The R.P again})$$

• Log(x) (Base e assumed)

$\log(0)$ is undefined \therefore expand $\log(1+x)$ about $x=0$ instead.

$$\text{if } f(x) \equiv \log(1+x) \quad f(0) = 0$$

$$f'(x) = \frac{1}{1+x}, \quad f'_0 = 1$$

$$f''(x) = -\frac{1}{(1+x)^2}, \quad f''_0 = -1$$

$$f'''(x) = \frac{2}{(1+x)^3}, \quad f'''_0 = 2$$

In fact Euler Showed that

13.19

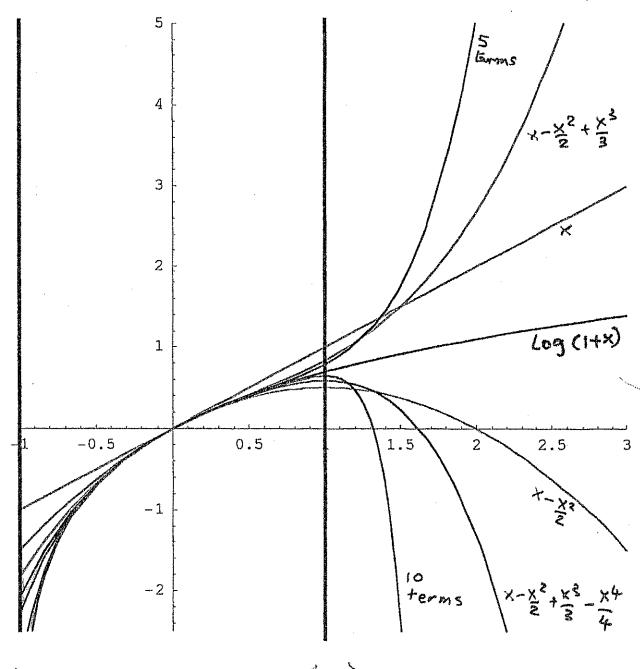
13.2

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} - \log n \right)$$

$$\rightarrow \gamma = 0.577215665 \dots$$

"Euler's Constant"

Comparison of $\log(1+x)$ and sum of first few terms in power series



Range of
Convergence
 $|x| < 1$

Agreement
WORSE
as add more
terms in
divergent region

13.5 Limits for Large & Small x.

13.21

We can use our power series and/or Taylor series to get limits of expressions as $x \rightarrow 0$ or $x \rightarrow \infty$.

Examples

$$\bullet \frac{\sin x}{x} \rightarrow 1 \text{ as } x \rightarrow 0$$

$$\text{for } |x| \ll 1 \quad \sin x \sim x - \frac{x^3}{3!} + O(x^5)$$

$$\therefore \frac{\sin x}{x} \sim 1 - \frac{x^2}{3!} + \dots \\ \rightarrow 1 \text{ as } |x| \rightarrow 0.$$

The $\text{Sinc}(x) f^n$ is defined as

$$\text{Sinc}(0) = 1$$

$$\text{Sinc}(x) = \frac{\sin x}{x} \quad x \neq 0$$

∴ It is continuous & well behaved.

$$\begin{aligned} \frac{d}{dx}(\text{Sinc}(x)) &= \frac{x \cos(x) - \sin(x)}{x^2} \\ &= \left(x - \frac{x^3}{2!} + \frac{x^5}{4!} - \dots - x + \frac{x^3}{3!} - \frac{x^5}{5!} \right) / x^2 \\ &= \left(\frac{1}{6} - \frac{1}{2} \right) x + O(x^3) \rightarrow 0 \text{ as } x \rightarrow 0 \text{ etc.} \end{aligned}$$

1.6 Newton Raphson Method

13.23

Use to solve transcendental equations (or cubics etc.)

$$\text{eg } x^2 = \log(5+x)$$

$$\text{write as } f(x) = x^2 - \log(5+x) = 0$$

$$\text{guess a trial soln, } x = k_0 \quad (\text{eq } k_0 = 2)$$

If k_{act} is the actual root:

$$0 = f(k_{\text{act}} + h) = f(k_0) + h f'_{k_0} + \dots$$

$$\begin{aligned} \text{Neglect terms of } O(h^2) &\quad \text{Remember this is } O(h^2) \\ \Rightarrow h &= -\frac{f(k_0)}{f'_{k_0}} \end{aligned}$$

$$\therefore \text{try } k_1 = k_0 - \frac{f(k_0)}{f'_{k_0}} \quad \text{as a}$$

better estimate
& iterate.

$$\text{if } f(x) = x^2 - \log(5+x)$$

$$f'(x) = 2x - \frac{1}{5+x} \quad \text{etc.}$$

$$\bullet \underline{(x^4 - x^2)^{\frac{1}{4}}} \text{ as } x \rightarrow \infty ?$$

Trick expand in powers of $\frac{1}{x}$ since $\frac{1}{x} \rightarrow 0$ as $x \rightarrow \infty$

$$(x^4 - x^2)^{\frac{1}{4}} = (x^4)^{\frac{1}{4}} \left(1 - \frac{1}{x^2}\right)^{\frac{1}{4}}$$

$$= x \left(1 - \frac{1}{x^2}\right)^{\frac{1}{4}}$$

$$= x \left\{ 1 - \frac{1}{4} \frac{1}{x^2} + \frac{1}{4} \cdot \frac{-3}{4} \frac{1}{2!} \left(\frac{-1}{x^2}\right)^2 \right.$$

$$\left. + \frac{1}{4} \cdot \frac{-3}{4} \cdot \frac{7}{4} \frac{1}{3!} \left(\frac{-1}{x^2}\right)^3 \dots \right\}$$

$$\rightarrow x \left(1 - \frac{1}{4} \frac{1}{x^2} + \dots\right)$$

$$\rightarrow x - \frac{1}{4} \frac{1}{x} + \dots$$

• L'Hopital's Rule.

$$\lim_{x \rightarrow 0} \frac{f(x)}{g(x)} = \frac{f(0) + x f'_0 + \dots}{g(0) + x g'_0 + \dots}$$

$$\rightarrow \boxed{\frac{f'_0}{g'_0}} \quad \text{if } f(0) = g(0) = 0$$

Could have done $\frac{\sin(x)}{x}$ this way.

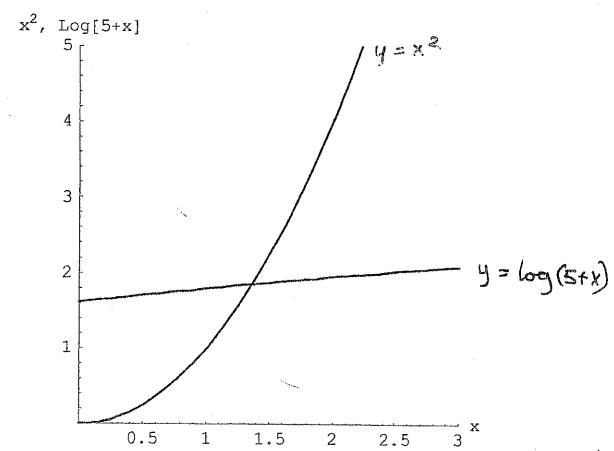
L15

13.24

Example of Newton's method for solving $x^2 = \log(5+x)$, starting at $k_0 = 2$

$$f(k) = x^2 - \log(5+x) \quad f'(k) = 2x - \frac{1}{5+x}$$

k	f(k)	f'(k)	h = -f(k)/f'(k)
2.0	2.05409	3.857143	-0.53254181
1.467458	0.286650	2.780296	-0.10310066
1.364358	0.010758	2.571590	-0.00418347
1.360174	1.77E-05	2.563120	-6.9125E-06
1.360167	4.84E-11	2.563106	-1.8873E-11



15 Hyperbolic Functions

15.1

15.

First recall:

$$e^{i\theta} = \cos \theta + i \sin \theta$$

$$e^{-i\theta} = \cos \theta - i \sin \theta$$

hence

$$\begin{aligned}\cos \theta &= \frac{e^{i\theta} + e^{-i\theta}}{2} \\ \sin \theta &= \frac{e^{i\theta} - e^{-i\theta}}{2i}\end{aligned}$$

Sometimes useful & show sin & cos are very similar to exponential

Put $x = i\theta$, hence $\theta = \frac{x}{i} = -ix$

$$\cos(ix) = \frac{e^x + e^{-x}}{2} = \cosh(x)$$

$$\sin(ix) = i \cdot \frac{e^x - e^{-x}}{2} = i \sinh(x)$$

where we have used $\cos(-ix) = \cos(ix)$
 $\& \sin(-ix) = -\sin(ix)$

The functions

$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$

$$\& \sinh(x) = \frac{e^x - e^{-x}}{2}$$

Depend only on the real variable x and have properties somewhat analogous to sin & cos.

- One of the main uses of sinh & cosh is for doing integrals.

Make more analogies:

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

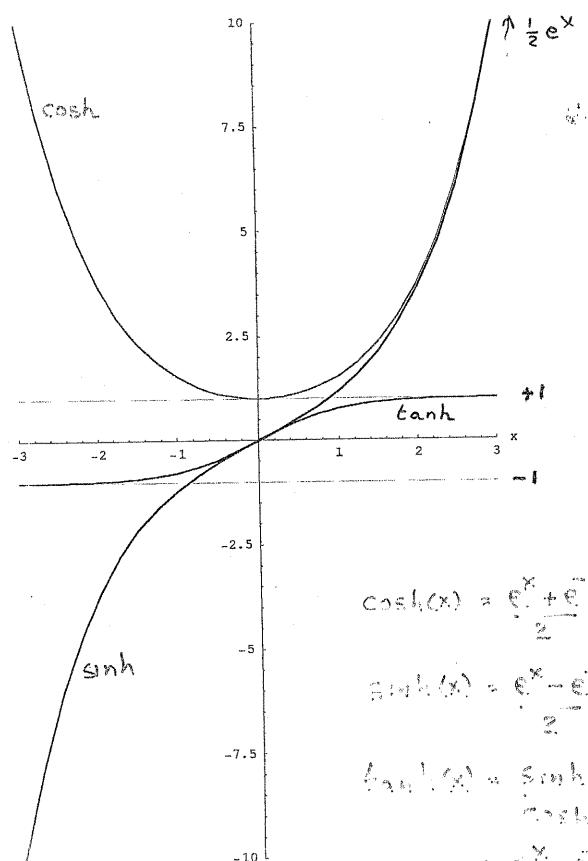
$$\operatorname{sech}(x) = \frac{1}{\cosh(x)}$$

$$\operatorname{csch}(x) = \frac{1}{\sinh(x)}$$

$$\operatorname{coth}(x) = \frac{1}{\tanh(x)}$$

these are
mainly
used.

Hyperbolic Functions



15.3

15.4

There exist many formulae analogous to the corresponding trig ones:

$$\text{eg } \cos^2 x + \sin^2 x = 1$$

$$\cosh^2 x - \sinh^2 x = 1$$

$$\sin(a+b) = \sin a \cos b + \cos a \sin b$$

$$\begin{aligned}\sinh(a+b) &= \sinh(a)\cosh(b) \\ &\quad + \cosh(a)\sinh(b)\end{aligned}$$

$$\tan(a+b) = \frac{\tan(a) + \tan(b)}{1 - \tan(a)\tan(b)}$$

$$\tanh(a+b) = \frac{\tanh(a) + \tanh(b)}{1 + \tanh(a)\tanh(b)}$$

Osborn's Rule is useful for remembering these.

"Flip the sign of any \sin^2 terms to get hyperbolic"

$$\cosh(\sinh x) = \sinh(\cosh x)$$

Osborn's rule just comes from our formulae for $\cos(cx)$ & $\sin(cx)$

15.5

= Locus

$$\begin{cases} \text{if } x = a \cos \theta \\ y = a \sin \theta \end{cases}$$

$$x^2 + y^2 = a^2$$

} The locus of (x, y) is a circle as θ varies from $0 \rightarrow 2\pi$

eg $\cos^2 \alpha + \sin^2 \alpha = 1$

but $\alpha = ix$

$$\cos^2(ix) + \sin^2(ix) = 1$$

$$\cosh^2 x + (\sinh x)^2 = 1$$

$$\cosh^2 x - \sinh^2 x = 1$$

etc.

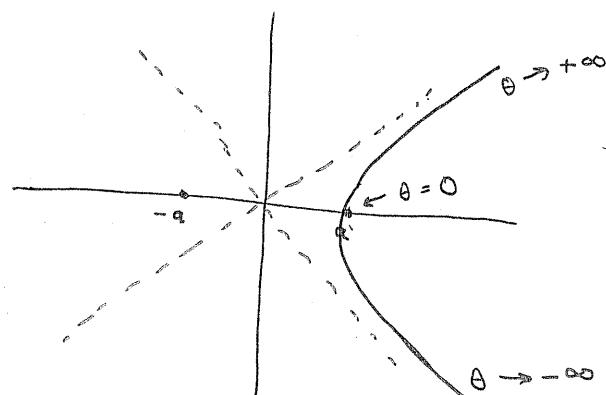
If have single sin factor in each term the factors of i cancel.

Note the hidden $\sin(a) \sin(b)$ in the denominator of our formula for $\tan(a+b)$

$$\begin{cases} \text{if } x = a \cosh \theta \\ y = a \sinh \theta \end{cases}$$

$$x^2 - y^2 = a^2$$

} The locus of (x, y) is a hyperbola as $\theta \rightarrow -\infty \rightarrow \infty$



Hence Name

Hyperbolic Functions

15.2 Inverse Hyperbolic Functions

15.7

15.8

define $y = \cosh^{-1} x$ by $\cosh(y) = x$
and $y = \sinh^{-1} x$ by $\sinh(y) = x$

(by analogy with sin & cos)

We can find formulae for these inverse fns:

$$\text{eg } x = \sinh y = \frac{e^y - e^{-y}}{2}$$

$$\Rightarrow 2x = e^y - e^{-y}$$

$$\Rightarrow (e^y)^2 - 2x e^y - 1 = 0$$

$$\Rightarrow e^y = \frac{2x \pm \sqrt{4x^2 + 1}}{2}$$

$$= x \pm \sqrt{x^2 + 1}$$

Must choose + since $e^y > 0$ (x real)

$$\Rightarrow y = \sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$$

Similarly

$$\cosh^{-1} x = \ln(x \pm \sqrt{x^2 - 1})$$

+ sign is principal value

$$\begin{aligned} \therefore \frac{d}{dx}(\sinh^{-1} x) &= \frac{d}{dx}(\ln(x + \sqrt{x^2 + 1})) \\ &= \frac{1 + \frac{x}{\sqrt{x^2 + 1}}}{x + \sqrt{x^2 + 1}} \\ &= \frac{1}{\sqrt{x^2 + 1}} \end{aligned}$$

$$\therefore \int \frac{dx}{\sqrt{x^2 + 1}} = \sinh^{-1} x$$

& hence $\int \frac{dx}{\sqrt{x^2 + a^2}} = \sinh^{-1} \frac{x}{a}$

Similarly

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1} \frac{x}{a} \quad x > a,$$

C.F. $\int \frac{dx}{\sqrt{a^2 - x^2}} = -\cos^{-1} \frac{x}{a} \quad |x| < a,$
or $= \sin^{-1} \frac{x}{a} \quad |x| < a,$

Finally $\tanh^{-1} x = \frac{1}{2} \ln \left(\frac{1+x}{1-x} \right)$

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{a} \tanh^{-1} \frac{x}{a} \quad |x| < a.$$

16 Ordinary Differential Equations

This is a BIG Topic!

6.1 Introduction

The basic ideas in physics & chemistry often concern the effect of something (eg force) on something else (eg point mass). This causes a **change** which results in a differential equation

Force = rate of change of momentum

$$F = m\ddot{a}$$

$$\Rightarrow F = m \frac{d^2x}{dt^2} \quad \text{Newton's Law once more.}$$

It could be argued that the chief aim of these maths courses is to equip you to solve differential eqns!

The above equations are all **ordinary** differential equations. The unknown function, y or ψ or I depends only on a **single** variable (x or t)

Hence only $\frac{dy}{dx}$ or $\frac{d\psi}{dx}$ appear - i.e. NO partial derivatives.

Later you will meet **partial** differential equations

$$\frac{\partial^2 f}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2} \quad f = f(x, t) \quad \text{Wave Eqn}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{\alpha^2} \frac{\partial u}{\partial t} \quad \text{Diffusion Equation.}$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \quad \text{Laplace's Eqn.}$$

and many more

A zoo of differential equations:

$$\bullet \frac{d^2 y}{dt^2} = -\omega^2 y$$

Simple Harmonic motion (SHM)

$$\bullet \frac{dy}{dt} = \lambda y$$

exponential growth ($\lambda > 0$) or decay ($\lambda < 0$)

$$\bullet \frac{dy}{dt} = r y \left(1 - \frac{y}{K}\right)$$

Logistic Eqn for population growth.

$$\bullet -\frac{\hbar^2}{2m} \frac{d^2 \psi}{dx^2} + V(x) \psi = E \psi$$

1D Schrödinger Eqn for wavefn ψ & energy E of particle in potential well $V(x)$

$$\bullet L \frac{d^2 I}{dt^2} + R \frac{dI}{dt} + \frac{I}{C} = \frac{dV}{dt}$$

Current I in series circuit driven by variable voltage V .

16.3

The **order** of a differential eqn is the highest derivative that occurs.

thus

$$\frac{dy}{dt} = \lambda y \quad \text{is } 1^{\text{st}} \text{ order}$$

and

$$\frac{d^2 y}{dt^2} = -\omega^2 y \quad \text{and}$$

$$L \frac{d^2 I}{dt^2} + R \frac{dI}{dt} + \frac{I}{C} = \frac{dV}{dt}$$

are both 2nd order

To solve differential eqns we have to integrate. Thus we might not be able to find an analytical soln.

(we can always find numerical solns in particular cases).

Also we use a series of tricks to do the integration - there is no methodic recipie.

16.2

Further Jargon - there are a number of other specialized terms used to describe ODE's (Ordinary Differential Equations). Eg for:

$$\left(\frac{d^3y}{dx^3}\right)^2 + xy^2 \frac{dy}{dx} + x^2y = \cos(x)$$

y is the dependant variable

x is the independant variable

(y & all derivatives are fns of x)

This eqn is of 2nd Degree because the power of the highest derivative is 2 (after removing fractional powers if any).

This eqn is NON-LINEAR because powers of y and/or derivatives different from 1 occur. (A term like $y\frac{dy}{dx}$ is also non-linear).

16.6

and $\frac{d^n}{dx^n}(A_1 + A_2x + A_3x^2 + \dots + A_nx^{n-1}) = 0$

so we "lose" n pieces of information in constructing $\frac{d^ny}{dx^n}$.

Notice the constants A_1, \dots, A_n each multiply a different power of x - thus they are genuinely different, it is no good just replacing A_1 by $B+C$ and claiming $B+C$ are independent.

The n arbitrary constants can be used to satisfy n different constraints on the general solution, for example the value of y at n different x values or the value of y, $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$... $\frac{d^{n-1}y}{dx^{n-1}}$ at a single point $x=a$.

Or we could have a mixture

The equation

$$x^2 \frac{d^2y}{dx^2} + 7y = \cos(x)$$

is a 2nd order linear ODE of degree 1. Y is the dependent variable, x is the independant variable.

In practice the ODE's that you need to study are usually of degree 1 and quite often linear.

16.2 Arbitrary Constants in Soln.

An ODE of order n will have a general solution containing n arbitrary constants.

This is reasonable, the equation will have a term with $\frac{dy}{dx^n}$

16.7

Some times ODE's also have singular solutions. These are solutions with no arbitrary constants that can't be got directly from the general soln. Cf L8 on question sheet.

16.3 FIRST ORDER ODES

The general case is

$$\frac{dy}{dx} = F(x, y)$$

where F is some general function of x & y. Even this can only be solved analytically in some cases.

The nature of F determines the "method" we choose in attempting to solve this. We will look at 4 cases:

16.3.1 Variables Separable

16.8

If $F(x,y) = f(x)g(y)$ we have

$$\frac{dy}{dx} = f(x)g(y)$$

$$\therefore \frac{dy}{g(y)} = f(x)dx$$

$$\text{i.e. } \int \frac{dy}{g(y)} = \int f(x)dx + K$$

is general soln with arb. const. K

PROVIDED we can do the integrals

Examples

(a) Exponential Growth

$$\frac{dy}{dx} = \lambda y$$

$$\frac{dy}{y} = \lambda dx \Rightarrow \ln y = \lambda x + K$$

$$y = e^{\lambda x + K} = e^K e^{\lambda x} = y_0 e^{\lambda x}$$

where $y = y_0$ at $x=0$.

b) Logistic Equation

16.10

This is a better model for population growth:

$$\frac{dy}{dt} = r y \left(\frac{K-y}{K} \right) = r y \left(1 - \frac{y}{K} \right)$$

• r is the "rate constant" it determines the initial exponential growth when y is very small.

• K is the "carrying capacity" we will see that it is the limiting population at large t .

• The $-r y^2 / K$ term represents death due to spread of disease or competition for food. Each member of the population has an effect on all the other members, hence y_2 .

• The eqn is separable and quite easy to solve

NB if the constant $\lambda > 0$ then $y \rightarrow \infty$ as $x \rightarrow \infty$

if $\lambda < 0$ $y \rightarrow 0$ as $x \rightarrow \infty$.

- Often we use time t rather than x as the independent variable - this, of course, makes no difference to the form of the soln, i.e.

$$\frac{dy}{dt} = \lambda y \Rightarrow y = y_0 e^{\lambda t}$$

- This is a bad model for most population growth predictions because a real population cannot $\rightarrow \infty$.
- Disease, Lack of food etc limit real populations
- Exponential growth is often a reasonable approx when a population is very small.

$$\frac{dy}{dt} = r y \left(1 - \frac{y}{K} \right)$$

$$\Rightarrow \frac{dy}{y \left(1 - \frac{y}{K} \right)} = r dt$$

$$\int \frac{dy}{y \left(1 - \frac{y}{K} \right)} = \int r dt = rt + C$$

$$\text{L.H.S.} = \int \frac{dy}{K} \left\{ \frac{K}{y} + \frac{1}{(1-y/K)} \right\}$$

$$= \frac{1}{K} (K \ln y - K \ln (1-y/K))$$

$$= \ln \left(\frac{y}{1-y/K} \right)$$

$$\therefore \ln \frac{y}{1-y/K} = rt + \ln \left(\frac{y_0}{1-y_0/K} \right)$$

$$\frac{y}{y_0} \left(1 - \frac{y_0}{K} \right) = \left(1 - \frac{y}{K} \right)^{rt}$$

$$y e^{-rt} \left(1 - \frac{y_0}{K} \right) = y_0 \left(1 - \frac{y}{K} \right)$$

$$y \left[\frac{y_0}{K} + e^{-rt} \left(1 - \frac{y_0}{K} \right) \right] = y_0$$

$$y = \frac{K}{1 + \left(\frac{K-y_0}{y_0} \right) e^{-rt}}$$

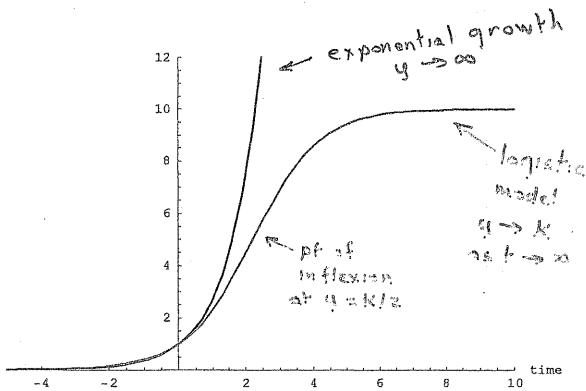
$y \rightarrow K$
as $t \rightarrow \infty$

Comparison of Exponential and Logistic population growth models

Population = 1 at time t = 0

Rate Constant r = 1

Carrying Capacity K = 10



$$\frac{dy}{dt} = ry \quad \text{Exponential}$$

$$\frac{dy}{dt} = ry - \frac{ry^2}{K} \quad \text{Logistic}$$

16.12

(c) Rearrange to get separable eqn.

Sometimes we can rearrange or substitute to get a separable eqn: eg -

$$\begin{aligned} \frac{dy}{dx} &= x + xy & ? \quad \text{Not a product} \\ &= x(1+y) & \text{Now is a product} \end{aligned}$$

$$\frac{dy}{1+y} = x dx$$

$$\ln(1+y) = \frac{1}{2}x^2 + C$$

$$1+y = e^{\frac{1}{2}x^2+C} = (1+y_0)e^{x^2/2}$$

$$y = y_0 \text{ at } x=0.$$

16.3.2 Homogeneous ODE

If $F(x, y) = f\left(\frac{y}{x}\right)$ ie

$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$ the eqn is
HOMOGENEOUS

If we put $\frac{y}{x} = v(x)$ where $v(x)$ is

so function of x then $y = xv(x)$

hence $\frac{dy}{dx} = 1.v(x) + x\frac{dv}{dx}$

ODE \rightarrow

$$x\frac{dv}{dx} + v = f(v) \quad \text{Separable!}$$

$$\frac{dv}{f(v)-v} = \frac{dx}{x}$$

$$\text{hence } \int \frac{dv}{f(v)-v} = \log x + A$$

Still have to do the integral on LHS.

16.13

2 Examples:

$$(a) \quad x\frac{dy}{dx} = x^2 + y^2$$

All terms 2nd order in x & $y \Rightarrow$ homogeneous

$$\frac{dy}{dx} = \frac{x}{y} + \frac{y}{x}$$

$$\text{put } y = vx, \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = \frac{1}{v} + v$$

$$v dv = \frac{dx}{x}$$

$$\frac{1}{2}v^2 = \ln x + K = \ln(Ax)$$

$$v = \pm \sqrt{2\ln(Ax)}$$

$$y = \pm x\sqrt{2\ln(Ax)}$$

$$(b) \quad \frac{dy}{dx} = \frac{x+y+1}{x+2y+1}$$

Not yet homogeneous - try substitution
 $u = x+1$ (easy to spot in this case)

put $u = x + 1$

$$\text{then } du = dx \quad \& \frac{dy}{du} = \frac{dy}{dx} \frac{dx}{du} = \frac{dy}{dx}$$

$$\therefore \text{ODE} \rightarrow \frac{dy}{du} = \frac{u+y}{u+2y}$$

$$\text{Now put } y = vu, \quad \frac{dy}{du} = v + u \frac{dv}{du}$$

$$v + u \frac{dv}{du} = \frac{u+y}{u+2y} = \frac{1+v}{1+2v}$$

$$\therefore u \frac{dv}{du} = \frac{1+v - v - 2v^2}{1+2v}$$

$$\int \frac{(1+2v) dv}{1-2v^2} = \int \frac{du}{u}$$

↓

write denominator as

$$\frac{1}{2} \left(\frac{1}{1+\sqrt{2}v} + \frac{1}{1-\sqrt{2}v} \right) \quad \text{etc. deal with } \int \frac{1 dv}{1-2v^2}$$

$$\& \int \frac{2v dv}{1-2v^2} = -\frac{1}{2} \ln(1-2v^2)$$

in the limit we write this as:

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \quad \text{chain rule}$$

where

$\frac{\partial f}{\partial x}$ is partial derivative of f wrt. x

& $\frac{\partial f}{\partial y}$ is partial derivative of f wrt. y

You evaluate $\frac{\partial f}{\partial x}$ by differentiating wrt. x pretending that y is constant.

$$\text{eg } f(x,y) = x^3 + y^3 + 2xy^2$$

$$\text{then } \frac{\partial f}{\partial x} = 3x^2 + 2y^2$$

$$\text{and } \frac{\partial f}{\partial y} = 3y^2 + 4xy$$

16.15

16.3.3 Exact Differentials

$$\text{if } \frac{dy}{dx} = F(x,y) = -\frac{P(x,y)}{Q(x,y)}$$

we can write this as:

$$P(x,y) dx + Q(x,y) dy = 0$$

and it might turn out that there is some function $f(x,y)$ such that

$$df = P dx + Q dy = 0$$

in which case

$$f = \text{constant}$$

is required
soln

As we will see in chapter 18 the change δf in f due to changes δx & δy in x & y is:

$$\boxed{\delta f = \frac{\partial f}{\partial x} \delta x + \frac{\partial f}{\partial y} \delta y}$$

16.16

Thus comparing

$$df = P dx + Q dy$$

$$\& df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

we see that the original ODE is exact iff

$$\frac{\partial f}{\partial x} = P \quad \& \quad \frac{\partial f}{\partial y} = Q$$

These are partial differential equations with solutions:

$$f = \int P(x,y) dx + A(y)$$

$$\& f = \int Q(x,y) dy + B(x)$$

where $A(y)$ is an arbitrary fⁿ of y

& $B(x)$ is an arbitrary fⁿ of x

We have to pick $A(y)$ & $B(x)$ so that both answers give the same function f . We can do this if and only if $(i.e.)$ $\int \frac{\partial P}{\partial y} dx = \int \frac{\partial Q}{\partial x} dy$

Fortunately there is a simple test for exact differentials

$$\text{if } P = \frac{\partial f}{\partial x} = \text{some fn of } x \& y$$

$$\text{then } \frac{\partial P}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right)$$

$$\& \text{ if } Q = \frac{\partial f}{\partial y}$$

$$\text{then } \frac{\partial Q}{\partial x} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right)$$

But as we will see

$$\boxed{\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}}$$

One less thing to remember

For all functions of x & y

" $\frac{\partial}{\partial x}$ & $\frac{\partial}{\partial y}$ commute"

$$\boxed{\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}}$$

is TEST for exact differential

$$\therefore P = \frac{\partial f}{\partial x} = \sin y$$

$$\Rightarrow f = \int \sin y \, dx = x \sin y + A(y)$$

$$\text{and } Q = \frac{\partial f}{\partial y} = x \cos y - \sin y$$

$$\begin{aligned} \Rightarrow f &= \int (x \cos y - \sin y) \, dy \\ &= x \sin y + \cos y + B(x) \end{aligned}$$

These 2 expressions for $f(x,y)$ are consistent if

$$A(y) = \cos y$$

$$\& B(x) = 0 \quad (\text{or any other const})$$

$$\therefore f = x \sin y + \cos y$$

But the ODE said $\boxed{df=0}$...

the soln is $f = \text{constant}$

$$\boxed{x \sin y + \cos y = K}$$

Now check the answer satisfies the original ODE!

In our example:

$$f(x,y) = x^3 + y^3 + 2xy^2$$

$$\frac{\partial f}{\partial x} = 3x^2 + 2y^2, \quad \frac{\partial f}{\partial y} = 3y^2 + 4xy$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y} (3x^2 + 2y^2) = 4y$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} (3y^2 + 4xy) = 4y \quad \text{check!}$$

Example 2

$$\frac{dy}{dx} = \frac{-\sin y}{x \cos y - \sin y}$$

$$df = \boxed{\sin y \, dx + (x \cos y - \sin y) \, dy = 0}$$

$$\stackrel{?}{P} = \frac{\partial f}{\partial x} ?$$

$$\stackrel{?}{Q} = \frac{\partial f}{\partial y} ?$$

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y} (\sin y) = \cos y$$

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x} (x \cos y - \sin y) = \cos y$$

\therefore This is exact

Check

$$\text{if } x \sin y + \cos y = K.$$

diff wrt x

$$1. \sin y + x \cos y \frac{dy}{dx} - \sin y \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} (x \cos y - \sin y) = -\sin y$$

$$\frac{dy}{dx} = \frac{-\sin y}{x \cos y - \sin y} \quad \checkmark$$

Suppose $\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$ is NOT exact?

Sometimes we can spot an integrating factor $\mu(x,y)$ that makes the eqn exact, ie

$$\mu P \, dx + \mu Q \, dy = 0$$

is exact

$$\frac{\partial}{\partial y} (\mu P) = \frac{\partial}{\partial x} (\mu Q)$$

Actually an integrating factor μ usually exists (in fact more than one μ is not unique) BUT the eqn for μ might be too hard to solve. Sometimes the method works:

Example 2

$$2x \log x \frac{dy}{dx} + y = 0$$

$$\text{or } y dx + 2x \log x dy = 0$$

$$\begin{array}{l} \downarrow \\ P \\ \downarrow \\ \frac{\partial P}{\partial y} = 1 \end{array} \quad \begin{array}{l} \downarrow \\ Q \\ \downarrow \\ \frac{\partial Q}{\partial x} = 2 \log x + 2 \end{array}$$

∴ NOT EXACT

$$\text{Try } \mu = \frac{y}{x}$$

$$\begin{array}{l} \frac{y^2}{x} dx + 2y \log x dy = 0 \\ \downarrow \\ \frac{\partial P}{\partial y} = 2y \\ \downarrow \\ \frac{\partial Q}{\partial x} = \frac{2y}{x} \end{array}$$

∴ Exact

Example 2 Revisited

$$y dx + 2x \log x dy = 0$$

$$\text{Try } \mu = x^a y^b$$

$$\Rightarrow x^a y^{b+1} dx + \underbrace{2x^{a+1} y^b \log x dy}_P = 0$$

$$\frac{\partial P}{\partial y} = (b+1)x^a y^b$$

$$\frac{\partial Q}{\partial x} = 2(a+1)x^a y^b \log x + 2x^a y^b$$

This works if

$$a+1 = 0 \Rightarrow a = -1$$

$$\text{and } b+1 = 2 \Rightarrow b = 1$$

$$\therefore \mu = x^{-1} y^1 = \frac{y}{x}$$

This method can become quite hard work.

$$\therefore \frac{\partial f}{\partial x} = P = \frac{y^2}{x}$$

$$\Rightarrow f = \int \frac{y^2}{x} dx = y^2 \log x + A(y)$$

$$\text{AND } \frac{\partial f}{\partial y} = Q = 2y \log x$$

$$\Rightarrow f = \int 2y \log x dy = y^2 \log x + B(x)$$

$$\text{take } A=B=0; f = y^2 \log x$$

$$\text{sol'n is } \boxed{y^2 \log x = K}$$

Aside

This eqn is separable, you might like to try solving it directly. You will need to spot $\int \frac{1}{x \log x} = \log(\log x)$

But how did I know to choose $\mu = \frac{y}{x}$

One method is to try $\boxed{\mu = x^a y^b}$ for constants a & b and see what works. [A good method for Tripos Questions].

16.3.4 ODE's Linear in y

This is the 4th (& final) method

$$\boxed{\frac{dy}{dx} + S(x)y = R(x)}$$

is general 1st ORDER ODE LINEAR in y . [1st Degree also]

Provided you can integrate $S(x)$ these equations can always be done using the following trick:

$$\text{use } \boxed{\mu(x) = e^{\int S(x) dx}} \text{ as}$$

integrating factor

(NB $\int S(x) dx$ is the indefinite integral and is therefore a fⁿ of x)

i.e.

$$\begin{aligned} & e^{\int S(x) dx} \frac{dy}{dx} + y S(x) e^{\int S(x) dx} \\ & = R(x) e^{\int S(x) dx} \end{aligned}$$

The LHS is an exact differential, ie

$$\frac{d}{dx} \left[y e^{\int S(x) dx} \right]$$

$$= \frac{dy}{dx} e^{\int S(x) dx} + y S(x) e^{\int S(x) dx}$$

$$\therefore \frac{d}{dx} \left[y e^{\int S(x) dx} \right] = R(x) e^{\int S(x) dx}$$

or $y e^{\int S(x) dx} = \int R(x) e^{\int S(x) dx} dx$

This will make more sense when we do an example and replace $e^{\int S(x) dx}$ with an explicit function.

Also you have to be able to do the integrals!

Example

$$\frac{dy}{dx} + 2xy = x^2$$

Not separable
Not Homogeneous
Not exact
($P = 2xy - x^2$
 $R = 1$)

is of form

$$\frac{dy}{dx} + S(x)y = R(x)$$

$$\text{with } S(x) = 2x, R(x) = x^2$$

$$\therefore \text{multiply by } M(x) = e^{\int 2x dx} = e^{x^2}$$

$$e^{x^2} \frac{dy}{dx} + e^{x^2} 2xy = e^{x^2} x^2$$

$$\frac{d}{dx} \left[y e^{x^2} \right] = x^2 e^{x^2}$$

$$y e^{x^2} = \int x^2 e^{x^2} dx + C$$

This integral
we can't do

Analytically - but at least
we know this. Also we could
now get a numerical sol'n.

16.4 Summary of Methods for First Order ODE's

17 2nd Order Linear ODE's with Constant Coefficients

i.e. Equations of form:

$$a \frac{d^2y}{dx^2} + b \frac{dy}{dx} + cy = f(x) \dots (17.1)$$

where a, b, c are real constants & $f(x)$ is a real f^n of x .

It turns out that this eqn is important in many applications AND it is often easy to solve.

We will look at applications soon, but first some general results.

17.1 Complementary equation

The eqn

$$a \frac{d^2y}{dx^2} + b \frac{dy}{dx} + cy = 0 \dots (17.2)$$

is Linear & Homogeneous in y . It is called the complementary

$$\frac{dy}{dx} + S(x)y = R(x)$$

INTEGRATING FACTOR $e^{\int S(x) dx}$

- Finally Practice the examples. More

(The term homogeneous it being used here with a different meaning to our use in section 16.3 on 1st order ODEs).

- 17.2 is Linear which means that if $y_1(x)$ & $y_2(x)$ are each solns of the eqn - so is:

$$y = \lambda y_1(x) + \mu y_2(x)$$

where λ & μ are any constants.

This is easy to verify & is because $\frac{d}{dx}$ & $\frac{d^2}{dx^2}$ are "Linear operators", ie

$$\frac{d}{dx}(\lambda y_1 + \mu y_2) = \lambda \frac{dy_1}{dx} + \mu \frac{dy_2}{dx}$$

hence

$$\frac{d^2}{dx^2}(\lambda y_1 + \mu y_2) = \lambda \frac{d^2 y_1}{dx^2} + \mu \frac{d^2 y_2}{dx^2}$$

Because its an important idea we will spell it out:

- 17.2 is a 2nd order ODE, therefore the general soln will have 2 arbitrary constants.

Hence if y_1 & y_2 are any 2 independent (ie $\frac{y_1}{y_2} \neq \text{const}$) solns of 17.2

$$y = A y_1 + B y_2$$

is the general soln with 2 consts.

Adding solns in this way is called superposition

- We now **SOLVE** 17.2 by trying $y = e^{\alpha x}$ for $\alpha = \text{const}$

$$\frac{dy}{dx} = \alpha e^{\alpha x}, \quad \frac{d^2 y}{dx^2} = \alpha^2 e^{\alpha x}$$

$$\Rightarrow \alpha^2 e^{\alpha x} + b\alpha e^{\alpha x} + c e^{\alpha x} = 0$$

$$\Rightarrow \alpha^2 + b\alpha + c = 0; \quad \dots (17.3)$$

$$\begin{aligned} a \frac{d^2}{dx^2}(\lambda y_1 + \mu y_2) + b \frac{d}{dx}(\lambda y_1 + \mu y_2) \\ + c(\lambda y_1 + \mu y_2) \\ = a \lambda \frac{d^2 y_1}{dx^2} + b \lambda \frac{dy_1}{dx} + c \lambda y_1 \\ + a \mu \frac{d^2 y_2}{dx^2} + b \mu \frac{dy_2}{dx} + c \mu y_2 \\ = \lambda(a \frac{d^2 y_1}{dx^2} + b \frac{dy_1}{dx} + c y_1) \\ + \mu(a \frac{d^2 y_2}{dx^2} + b \frac{dy_2}{dx} + c y_2) \\ = \lambda 0 + \mu 0 = 0 \quad \text{QED.} \end{aligned}$$

Note this only works because the RHS of 17.2 is 0. If we tried this with 17.1 where the RHS is $f(x)$ we would get $(\lambda + \mu)f(x) \neq f(x)$ ie NOT another soln.

17.3 is a simple quadratic eqn for α and is called the auxiliary eqn. The roots are

$$\alpha_1, \alpha_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

hence solns of 17.2 are

$$y_1 = e^{\alpha_1 x} \quad \& \quad y_2 = e^{\alpha_2 x}$$

and the general soln is

$$y = A e^{\alpha_1 x} + B e^{\alpha_2 x}$$

where A & B are arbitrary consts.

17.2 Forms of Solutions to Aux Eqn

There are actually 3 cases to consider

α_1, α_2	- real	$b^2 - 4ac > 0$
	- complex	$b^2 - 4ac < 0$
	- equal	$b^2 - 4ac = 0$

17.2.1 Real Roots

17.6

 $\alpha_1, \alpha_2 = \alpha \pm \beta$ say where

$$\alpha = -\frac{b}{2a} \quad \beta = \sqrt{\frac{b^2 - 4ac}{2a}}$$

Note $|\beta| < |\alpha| \Rightarrow$ sign of α_1 & α_2
 depends only on sign of α

solⁿ is

$$y = A e^{(\alpha+\beta)x} + B e^{(\alpha-\beta)x}$$

$$= e^{\alpha x} [A e^{\beta x} + B e^{-\beta x}]$$

Simple exponential behaviour.

as $x \rightarrow \infty$ the $A e^{\beta x}$ term
 is $\gg B e^{-\beta x}$ term. (active)

However $y \rightarrow \infty$ if $\alpha > 0$ ie $b < 0 \quad a > 0$ or $y \rightarrow 0$ if $\alpha < 0$ ie $b > 0 \quad a > 0$ 17.2.2 Complex Roots

17.7

$$\text{get } \alpha_1, \alpha_2 = -\frac{b}{2a} \pm \frac{i\sqrt{4ac-b^2}}{2a}$$

$$= \alpha \pm i\gamma \text{ say}$$

hence have 2 solns:

$$y_1 = e^{\alpha x + i\gamma x} = e^{\alpha x} (\cos \gamma x + i \sin \gamma x)$$

$$y_2 = e^{\alpha x - i\gamma x} = e^{\alpha x} (\cos \gamma x - i \sin \gamma x)$$

Hence can take

$$\frac{y_1 + y_2}{2} = \boxed{e^{\alpha x} \cos \gamma x}$$

$$\& \frac{y_1 - y_2}{2i} = \boxed{e^{\alpha x} \sin \gamma x}$$

As indep solns. General Solⁿ:

$$y = e^{\alpha x} (A \cos \gamma x + B \sin \gamma x)$$

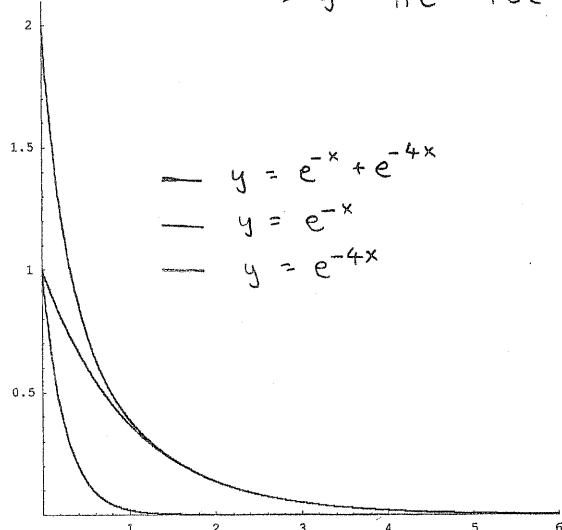
$$\text{or } y = e^{\alpha x} A' \cos(\gamma x + \delta)$$

$$\text{or } y = e^{\alpha x} B' \sin(\gamma x + \delta)$$

Solutions of Second Order ODE

$$y'' + 5y' + 4y = 0$$

$$\Rightarrow y = A e^{-x} + B e^{-4x}$$



The form of these solutions is thus a cosine with wavelength $\frac{2\pi}{\gamma}$ and amplitude that rises or falls like $e^{\alpha x}$ (depending on sign of b).

17.2.3 Equal rootsIf $b^2 = 4ac$ then

$$y = e^{\alpha x} \quad \text{where } \alpha = -\frac{b}{2a}$$

is a solution.

We need a second indep soln, which turns out to be

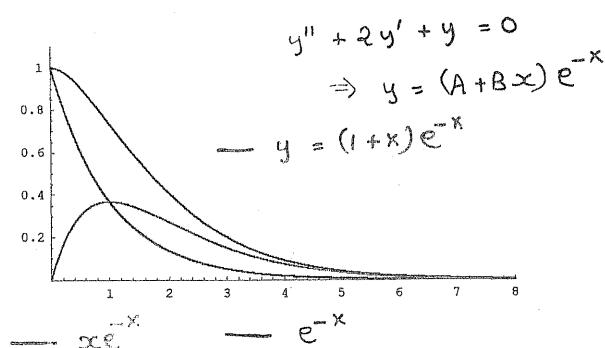
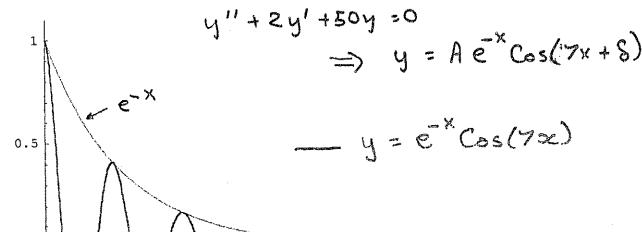
$$y = x e^{\alpha x}$$

Hence general solⁿ is

$$y = (A + Bx) e^{\alpha x} \quad \text{Learn This.}$$

Verification that $x e^{\alpha x}$ is a solⁿ is

Solutions of Second Order ODE



17.3 Examples

1) Solve $y'' + 5y' + 4y = 0$

where $y = 2$ & $y' = -5$ when $x = 0$

$$(y' = \frac{dy}{dx})$$

$$\text{Try } y = e^{\alpha x}$$

$$\Rightarrow \alpha^2 + 5\alpha + 4 = 0$$

$$\Rightarrow \alpha = -\frac{5}{2} \pm \frac{3}{2} = -1 \text{ or } -4$$

$$y = A e^{-x} + B e^{-4x} \text{ is gen soln.}$$

but $y = 2$ when $x = 0$

$$\Rightarrow A + B = 2$$

$$\& y' = -A e^{-x} - 4B e^{-4x} = -5 \text{ when } x = 0$$

$$\therefore -A - 4B = -5$$

$\therefore A = B = 1$ & required soln is

$$y = e^{-x} + e^{-4x}$$

2) Solve $y'' + 2y' + 50y = 0$

$$\text{Try } y = e^{\alpha x}$$

$$\Rightarrow \alpha^2 + 2\alpha + 50 = 0$$

$$\alpha = \frac{-2 \pm \sqrt{4 - 4 \cdot 50}}{2}$$

$$= -1 \pm i\sqrt{49} = -1 \pm 7i$$

\therefore Gen soln

$$y = A e^{-x} \cos(7x + \delta)$$

17.10 17.4 General Soln of eqn 17.1

It is now time to solve

$$ay'' + by' + cy = f(x) \dots \dots \dots (17.1)$$

$$y' = \frac{dy}{dx}$$

• If we can find any soln of 17.1

$$\text{say } y = Y(x)$$

• And the gen soln of

$$ay'' + by' + cy = 0 \dots \dots \dots (17.2)$$

$$\text{is } A y_1(x) + B y_2(x)$$

• Then

$$y = Y(x) + A y_1(x) + B y_2(x)$$

is the general soln of 17.1 Because

(a) 2 Arb. Constant

(b) soln satisfies 17.1.

Gen soln

$$y = (A + Bx) e^{-3x}$$

3) Solve $y'' + 6y' + 9y = 0$

$$\text{Try } y = e^{\alpha x}$$

$$\Rightarrow \alpha^2 + 6\alpha + 9 = 0$$

$$(\alpha + 3)^2 = 0 \Rightarrow \text{equal roots } \alpha = -3$$

Gen soln

$$y = (A + Bx) e^{-3x}$$

Y(x) (ie any specific solⁿ of 17.1) is called the "Particular Integral"

and A₁y₁(x) + A₂y₂(x) is called the "Complementary function" (and is easy to find).

The general solⁿ is obtained by adding the "P.I." to the "C.F."

We still need to find the P.I. - which obviously depends on what f(x) actually is.

A good method is to try and find a P.I. which is a similar fⁿ to f(x).

We will look at 3 cases:

$f(x)$ = simple polynomial
= exponential
= sin or cos.

17.14 (1) $F(x) = \underline{\text{simple polynomial}}$

• example

$$y'' + 5y' + 4y = 7$$

can guess $y = \frac{7}{4}$ will work.

hence general solⁿ is

$$\boxed{y = A e^{-4x} + B e^{-x} + \frac{7}{4}}$$

• Example

$$y'' + 5y' + 4y = x + 2$$

Now harder to guess ∴ try

$$\boxed{y = \alpha x + \beta}, \quad y' = \alpha \quad \boxed{\alpha \& \beta \text{ to be determined}}$$

$$\text{then } 5\alpha + 4(\alpha x + \beta) = x + 2$$

$$\text{or } 4\alpha x + (5\alpha + 4\beta) = x + 2$$

But this holds for all x values

$$\therefore 4\alpha = 1 \rightarrow \alpha = \frac{1}{4}$$

$$\& \quad 5\alpha + 4\beta = 2$$

$$\rightarrow 4\beta = 2 - \frac{5}{4} = \frac{3}{4} \therefore \beta = \frac{3}{16}$$

$$\text{gen sol}^n \quad \boxed{y = A e^{-4x} + B e^{-x} + \frac{x}{4} + \frac{3}{16}}$$

22 In general we can match up to 3 constants, ie if RHS is say

$$a_1 + a_2 x + a_3 x^2$$

we can try

$$y = b_1 + b_2 x + b_3 x^2$$

and get 3 sim. eqns for the 3 unknown b's.

$$(2) \quad F(x) = e^{\omega x}$$

Assuming ω is NOT a root of the auxilliary eqn (ie not part of the C.F.)

then try

$$y = C e^{\omega x}$$

$$\text{eg} \quad \boxed{y'' + 5y' + 4y = e^{2x}}$$

$$\text{try } y = C e^{2x}$$

$$\Rightarrow C 4e^{2x} + 10Ce^{2x} + 4Ce^{2x} = e^{2x}$$

$$\Rightarrow 18C = 1 \quad C = \frac{1}{18}$$

$$\text{gen sol}^n \quad \boxed{y = A e^{-4x} + B e^{-x} + \frac{1}{18} e^{2x}}$$

17.16

If ω IS a root of the auxilliary eqn try $y = C x e^{\omega x}$

$$\text{eg} \quad y'' + 5y' + 4y = e^{-x}$$

$$\text{try } y = C x e^{-x}$$

$$y' = C e^{-x} - C x e^{-x}$$

$$y'' = -C e^{-x} - C x e^{-x} + C x e^{-x}$$

$$\therefore C e^{-x} \{ -2 + x + 5(1-x) + 4x \} = e^{-x}$$

$$C \{ 3 \} = 1 \Rightarrow C = \frac{1}{3}$$

$$\therefore \boxed{y = A e^{-4x} + B e^{-x} + \frac{x}{3} e^{-x}}$$

is gen solⁿ.

If ω is a double root of aux eqn then $y = C x^2 e^{\omega x}$ will work etc.

$$(3) F(x) = \sin \omega x \text{ or } \cos \omega x$$

17.18

If RHS is $a_1 \cos \omega x + a_2 \sin \omega x$

$$\text{try } y = b_1 \cos \omega x + b_2 \sin \omega x$$

& equate coeffs of sin & cos

- can get a bit messy.

Example

$$y'' + 4y' + 5y = \cos 2x$$

$$\text{try } y = C \cos 2x + D \sin 2x$$

$$\text{then } y' = -2C \sin 2x + 2D \cos 2x$$

$$y'' = -4C \cos 2x - 4D \sin 2x$$

$$\therefore \cos 2x \{-4C + 8D + 5C\}$$

$$+ \sin 2x \{-4D - 8C + 5D\} = \cos 2x$$

$$8D + C = 1$$

$$-8C + D = 0 \Rightarrow D = 8C$$

$$\therefore C = \frac{1}{65} \quad D = \frac{8}{65}$$

$$y = A e^{-4x} + B e^{-x} + \frac{1}{65} \cos 2x + \frac{8}{65} \sin 2x$$

is gen solⁿ.

Erratum
the 4 & 5
are transposed
 \therefore The CF
should have
complex
roots

Finally if $F(x)$ is sum of functions

$$\text{eg } F(x) = e^{3x} + \sin 2x$$

Try sum of potential solⁿs, i.e.

$$y = C e^{3x} + D \sin 2x + E \cos 2x$$

etc.

Summary

$$F(x) = a_2 x^2 + a_1 x + a_0$$

$$\text{try } y = b_2 x^2 + b_1 x + b_0$$

$$F(x) = a \cos \lambda x + b \sin \lambda x$$

$$\text{try } y = c \cos \lambda x + d \sin \lambda x$$

$$F(x) = a e^{\omega x}$$

$$\text{try } y = b e^{\omega x}$$

(or $b x e^{\omega x}$ if ω root)

There is an alternate (better?) way of doing this using complex exponentials:

$$\text{if } y_1'' + 4y_1' + 5y_1 = \cos 2x$$

$$\text{let } y_2'' + 4y_2' + 5y_2 = \sin 2x$$

$$\text{then } z'' + 4z' + 5z = e^{2ix}$$

$$\text{where } z = y_1 + iy_2$$

$$\text{Hence try } z = Ce^{2ix}$$

$$-4C e^{2ix} + 8iCe^{2ix} + 5Ce^{2ix} = e^{2ix}$$

$$C(1+8i) = 1$$

$$C = \frac{1}{1+8i}$$

$$\therefore z = \frac{1}{1+8i} e^{2ix} = \frac{(1-8i)}{1+8^2} e^{2ix}$$

$$\therefore y_1 = \text{Re}(z) = \text{Re} \left(\frac{1}{65} - \frac{8i}{65} \right) (\cos 2x + i \sin 2x)$$

$$= \frac{1}{65} \cos 2x + \frac{8}{65} \sin 2x. \quad \text{Same as before}$$

also

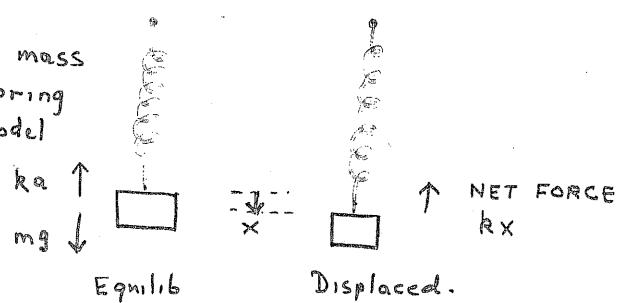
$$y_2 = \text{Im}(z) = \frac{1}{65} \sin 2x - \frac{8}{65} \cos 2x.$$

free bonus.

CF K7 & FG on question sheet.

17.5 Simple Harmonic Motion (SHM) 17.2

Take mass
on spring
as model



When displaced get

$$m\ddot{x} = -kx \quad \text{or} \quad \ddot{x} = -\frac{k}{m}x \quad \omega = \sqrt{\frac{k}{m}}$$

Write in our standard form

$$\boxed{\ddot{x} + \omega^2 x = 0}$$

Auxiliary eqⁿ $\alpha^2 + \omega^2 = 0 \Rightarrow \alpha = \pm i\omega$

$$\Rightarrow x = K e^{\pm i\omega t}$$

$$\text{or } x = A \cos(\omega t + \delta)$$

etc.

COMPLEX REPRESENTATION

It is often very convenient to think of

$$Z = |z| e^{i(\omega t + \delta)}$$

as the complex soln of

$$\ddot{Z} = -\omega^2 Z$$

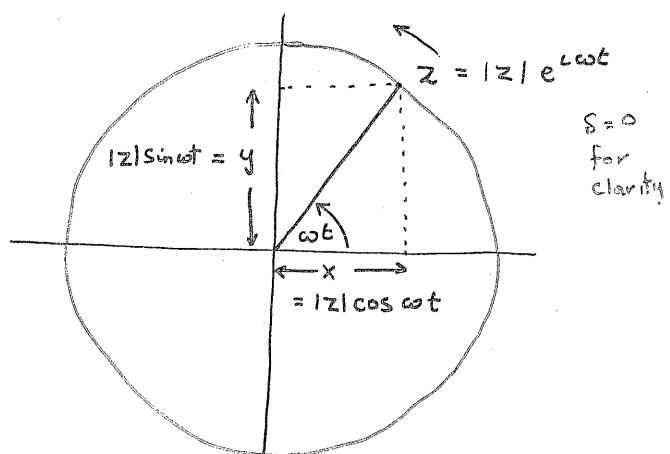
$$\& \operatorname{Re}(Z) = |z| \cos(\omega t + \delta)$$

As the required physical quantity.

Calculations are done using the complex representations of quantities until the end.

Then finally take Re part to get physical quantity.

In the Argand Diagram :



The complex representation Z rotates with angular velocity ω in the complex plane.

The projections on the real & imaginary axes perform SHM.

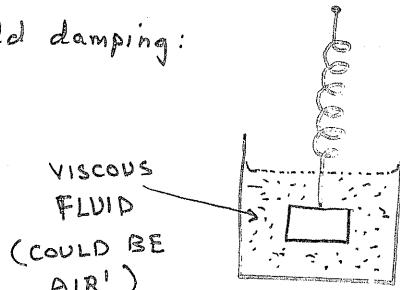
This picture is a good way to think about SHM.

17.6 Damped SHM

17.23

17.1

Add damping:



VISCOSUS DRAG
= $\lambda \dot{x}$
(i.e depends on Velocity)

$$m \ddot{x} = -kx - \lambda \dot{x}$$

or

$$\ddot{x} + \frac{\lambda}{m} \dot{x} + \omega^2 x = 0$$

we have already solved this with

$$\frac{b}{a} = \frac{\lambda}{m} \quad \& \quad \frac{c}{a} = \omega^2 \quad \text{Thus}$$

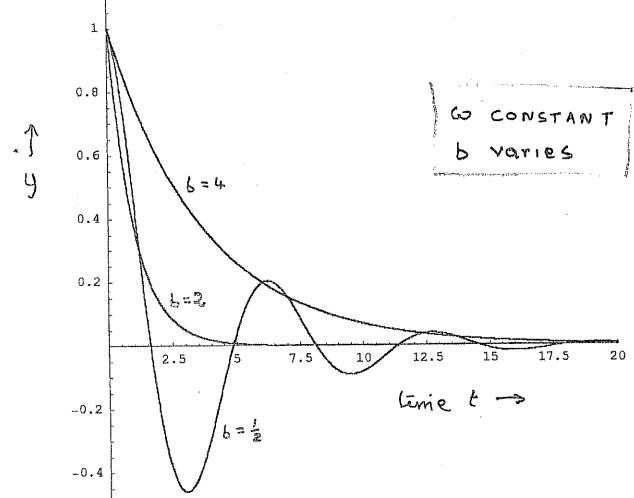
Real Roots: $(\frac{\lambda}{m})^2 > 4\omega^2$ Exponential Decay
"OVERDAMPING."

Complex Roots: $(\frac{\lambda}{m})^2 < 4\omega^2$ Damped oscillations
"UNDER DAMPING"

Equal Roots $(\frac{\lambda}{m})^2 = 4\omega^2$ "CRITICAL DAMPING"

Damped SHM

$$\frac{d^2y}{dt^2} + b \frac{dy}{dt} + \omega^2 y = 0$$



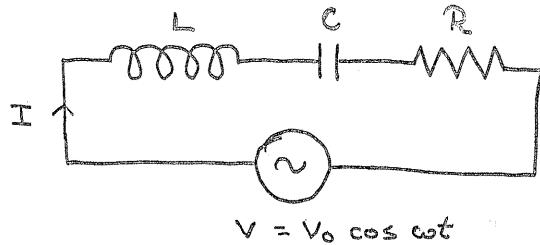
- underdamping $b = \frac{1}{2}$ $\omega = 1$
- - - critical damping $b = 1$ $\omega = 1$
- overdamping $b = 4$ $\omega = 1$

7.7 AC Circuits

17.25

[Not really for exam in this course]

The complex representation really comes into its own for AC Circuits



Consider $L, C \& R$ in series with AC voltage source $V = V_0 \cos \omega t$.

$$\begin{aligned} V &= V_L + V_C + V_R \\ &= L \dot{I} + \frac{Q}{C} + IR \end{aligned}$$

Voltage Drops
on each
component
 Q is charge
on C

But $I = \dot{Q}$ \therefore

$$L \ddot{Q} + R\dot{Q} + \frac{1}{C} Q = V_0 \cos \omega t$$

Damped, Driven SHM.

- The Complementary Function terms are called "Transients" and might be important in the design of real systems - but we are not interested.

To find P.I. of eqn use complex representation, i.e. solve

$$L \ddot{Q} + R\dot{Q} + \frac{1}{C} Q = V_0 e^{i\omega t}$$

& try $Q = Q_0 e^{i\omega t}$

$$\text{But } I = \dot{Q} = i\omega Q = I_0 e^{i\omega t}$$

$$\therefore Q = \frac{I}{i\omega} \quad \& \quad \ddot{Q} = \dot{I} = i\omega I$$

$$(i\omega L + R + \frac{1}{i\omega C}) I_0 e^{i\omega t} = V_0 e^{i\omega t}$$

$$I_0 = \left\{ \frac{V_0}{i\omega L + R + \frac{1}{i\omega C}} \right\} e^{i\omega t}$$

Pure complex number

• Comments on notation

ω is the applied frequency - not the natural freq which is $\frac{1}{\sqrt{LC}}$

We use capitals $I, V \& D$, engineers would use $i, v, \& q$ for AC quantities hence they use j for $\sqrt{-1}$ which is terrible.

• Solution of eqn

We have already done this. Assuming the circuit is turned on at time $t=0$ with random charge on C etc. This determines the 2 arbitrary constants in the Complementary Function. But the C.F. decays exponentially like $e^{-\frac{Rt}{2L}}$ hence after time $t \sim \frac{2L}{R} \times 3$ only the Particular Integral remains \rightarrow gives steady state of circuit.

17.27

This is just like Ohms Law

$$I = \frac{V}{R}$$

but with the complex impedance

$$Z = i\omega L + R + \frac{1}{i\omega C}$$

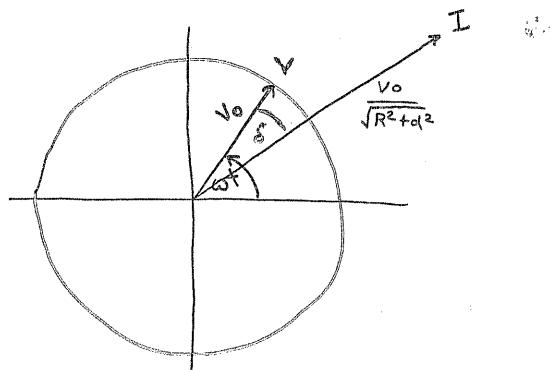
Replacing R .

$$\begin{aligned} \frac{1}{Z} &= \frac{1}{R + i(\omega L - \frac{1}{i\omega C})} = \frac{1}{R + i\alpha}, \quad \alpha = \omega L - \frac{1}{\omega C} \\ &= \frac{1}{R + i\alpha} \cdot \frac{R - i\alpha}{R - i\alpha} = \frac{R - i\alpha}{R^2 + \alpha^2} \\ &= \frac{\sqrt{R^2 + \alpha^2} e^{-i\delta}}{R^2 + \alpha^2} \quad \text{where} \\ &\quad \tan \delta = \frac{\alpha}{R} \\ &= \frac{P - iS}{\sqrt{R^2 + \alpha^2}} \end{aligned}$$

$$\therefore I = \frac{V_0}{\sqrt{R^2 + \alpha^2}} e^{i(\omega t - \delta)}$$

17.28

In the Argand diagram I & V rotate with angular vel ω and phase difference δ



- Current and Voltage "out of phase" by angle δ
- Current max at "Resonance freq"
 $\omega = \frac{1}{\sqrt{LC}}$ where $d=0$. ($\delta=0$)
- Physical Current is $\text{Re}(V)$
 $= \frac{V_0}{\sqrt{R^2+d^2}} \cos(\omega t - \delta)$
- Must take Re part first before working out power $= IV$

Hence

$$\frac{1}{Z} = \frac{1}{\omega L} + \frac{\omega C}{1} = \frac{1 - \omega^2 LC}{\omega L}$$

$$Z = \frac{\omega L}{1 - \omega^2 LC}$$

- Purely imaginary $\therefore 90^\circ$ phase change
- $|Z| \rightarrow \infty$ as $\omega^2 \rightarrow \frac{1}{LC}$

In practice $|Z|$ stays finite because there is always a small series R which we have ignored.

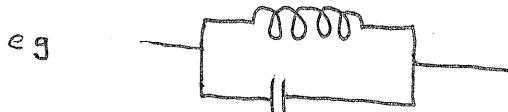
- This is how you tune a radio (at least on old fashioned one).

The above analysis was for a series circuit. However we can analyse any network by associating complex impedance

ωL with inductor L

$\frac{1}{\omega C}$ with capacitor C

R with resistance R (real)



Parallel L-C circuit has impedance:

$$\frac{1}{Z} = \frac{1}{\omega L} + \frac{1}{\omega C}$$

By analogy with

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

17.8 Beats

This is another simple application of complex representation

Consider

$$y_1 = \cos \omega_1 x + \cos \omega_2 x$$

where $\omega_1 - \omega_2$ is small.

i.e Superposition of 2 nearly equal cosine waves.

$$\text{if } y_1 = \cos \omega_1 x + \cos \omega_2 x$$

$$\text{let } y_2 = \sin \omega_1 x + \sin \omega_2 x$$

$$\text{then } Z = y_1 + iy_2 = e^{i\omega_1 x} + e^{i\omega_2 x}$$

$$= e^{i\frac{\omega_1 + \omega_2}{2}x} \left\{ e^{i\frac{\omega_1 - \omega_2}{2}x} + e^{-i\frac{\omega_1 - \omega_2}{2}x} \right\}$$

$$= 2e^{i\omega x} \cos \Delta \omega x$$

where $\omega = \frac{\omega_1 + \omega_2}{2}$ is mean ω

& $\Delta \omega = \frac{\omega_1 - \omega_2}{2}$ is $\frac{1}{2}$ difference

Hence

$$y_1 = \operatorname{Re}(z)$$

$$y_1 = 2 \cos \omega x \cos \Delta\omega x$$

Oscillations at the average ω

(wavelength $\lambda = \frac{2\pi}{\omega}$)

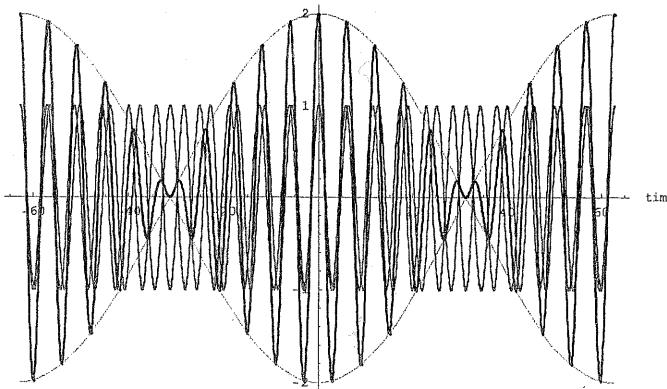
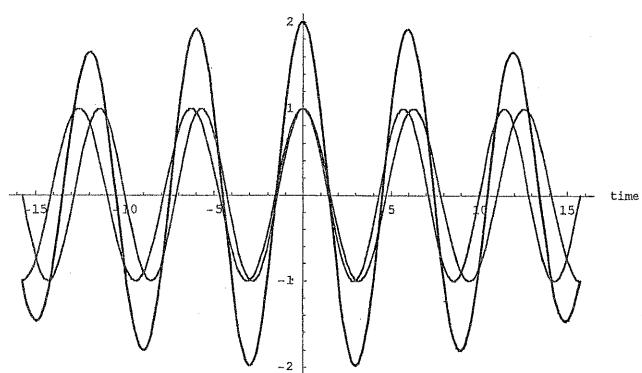
Modulated at wavelength $\frac{2\pi}{\Delta\omega}$
which is big.

⇒ Low freq "beating" you hear
tuning musical instruments.

[We clearly could have done this calc just using trig identities
but I think use of complex repn is much clearer]

P.S. This is essentially all the maths you need to understand neutrino oscillations]

Beats: $\cos(x) + \cos(1.1x)$



18.1 Introduction to Partial Derivatives

18.1 Chain Rule

Consider a fn of 2 variables $f(x, y)$

For the time being think of x & y as quite independent of each other.

For any fixed value of y , say y_0 , $f(x, y_0)$ is a simple fn of x alone & we can work out the derivative wrt x :

$$\frac{\partial f}{\partial x} = \lim_{\delta x \rightarrow 0} \frac{f(x+\delta x, y_0) - f(x, y_0)}{\delta x}$$

We use $\frac{\partial f}{\partial x}$ not $\frac{df}{dx}$ to indicate a partial derivative.

$\frac{\partial f}{\partial x}$ means diff wrt x treating y as a constant.

Similarly

$$\frac{\partial f}{\partial y} = \lim_{\delta y \rightarrow 0} \frac{f(x_0, y+\delta y) - f(x_0, y)}{\delta y}$$

$\frac{\partial f}{\partial y}$ means diff. wrt. y keeping x const

Example

$$f(x, y) = \sin(x) \cos^2(y)$$

$$\frac{\partial f}{\partial x} = \cos(x) \cos^2(y)$$

$$\frac{\partial f}{\partial y} = -2 \sin(x) \sin(y) \cos(y)$$

Suppose BOTH x & y change - how does f vary?

$$\delta f = f(x+\delta x, y+\delta y) - f(x, y)$$

$$= f(x+\delta x, y+\delta y) - f(x, y+\delta y) \\ + f(x, y+\delta y) - f(x, y)$$

$$\begin{aligned} \Delta f &= \frac{f(x+\delta x, y+\delta y) - f(x, y)}{\delta x} \cdot \delta x \\ &\quad + \frac{f(x, y+\delta y) - f(x, y)}{\delta y} \cdot \delta y \\ &\rightarrow \frac{\partial f}{\partial x} \delta x + \frac{\partial f}{\partial y} \delta y \\ &\quad \text{as } \delta x \text{ & } \delta y \rightarrow 0 \end{aligned}$$

[Strictly $\frac{\partial f}{\partial x}$ is evaluated at x & $y+\delta y$ but this is an error that tends $\rightarrow 0$ as $\delta x, \delta y \rightarrow 0$ i.e it is 2nd order in small quantities]

$$\Delta f = \frac{\partial f}{\partial x} \delta x + \frac{\partial f}{\partial y} \delta y$$

or
$$\frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}$$
 | Chain Rule.
in the limit

Also if $x = x(t)$ and $y = y(t)$,
i.e the both depend on some parameter t

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}$$
 "Divide by $\frac{dt}{dt}$ "

Example:

$$f(x, y) = x + y^2$$

$$\frac{\partial f}{\partial x} = 1 \quad \frac{\partial f}{\partial y} = 2y$$

$$\therefore df = 1 \cdot dx + 2y dy$$

$$\text{If } y = \sin^2 x \text{ say}$$

$$\frac{dy}{dx} = 2 \sin x \cos x$$

$$\begin{aligned} \frac{df}{dx} &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \frac{dy}{dx} = 1 + 2y \cdot 2 \sin x \cos x \\ &= 1 + 4 \sin^3 x \cos x \end{aligned}$$

This can also be written as

$$\boxed{\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \frac{dy}{dx}} \quad \text{"Divide by } \frac{dx}{dx} \text{"}$$

which actually means y is really a function of x , i.e $y = y(x)$ and $\frac{df}{dx}$ is the overall rate of change of f w.r.t. x due to the change in the explicit x dependence and the implicit x dependence of y .

$$\text{i.e } f(x, y) = f(x, y(x))$$

Similarly:

$$\boxed{\frac{df}{dy} = \frac{\partial f}{\partial x} \frac{dx}{dy} + \frac{\partial f}{\partial y}} \quad \text{"Divide by } \frac{dy}{dy} \text{"}$$

$$\text{if } x = x(y), f = f(x(y), y)$$

Of course we could have got this directly by

$$f = x + y^2 = x + \sin^4 x$$

$$\frac{df}{dx} = 1 + 4 \sin^3 x \cos x.$$

- Sometimes y might not be easy to substitute for

e.g if
$$y + \log y = x$$

then
$$\frac{dy}{dx} + \frac{1}{y} \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{y}{1+y}$$

so if $f = x + y^2$

$$\frac{df}{dx} = 1 + 2y \frac{dy}{dx} = 1 + \frac{2y^2}{1+y}$$

This might well be useful for example $y=1$ when $x=1$.

8.2 Second order Partial Derivatives 18.7

$\frac{\partial f}{\partial x}$ is in general also a f^n of x and y .
thus we can work out its partial derivatives:

$$\frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial x^2} \quad \text{2nd order partial derivative wrt } x.$$

$$\& \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial y \partial x}$$

Similarly for $\frac{\partial f}{\partial y}$:

$$\frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial x \partial y}$$

$$\frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial y^2}$$

We will now show

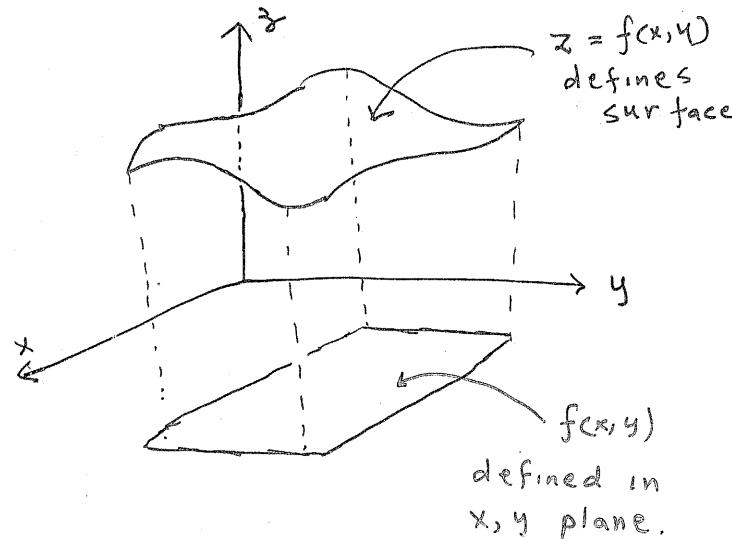
$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

i.e. you don't have to remember the order you did the differentiations.

[Strictly all the derivatives must exist and be continuous]

8.3 Visualisation of 2D functions 18.9

We can visualise $f(x, y)$ as the height, z , of a 2D undulating surface in 3D space:



We can now draw contour maps and look for peaks & troughs and so on.

→ Gives useful information.

Proof that $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$

$$\frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) = \lim_{\delta y \rightarrow 0} \frac{\frac{\partial f}{\partial x} \Big|_{x, y+\delta y} - \frac{\partial f}{\partial x} \Big|_{x, y}}{\delta y}$$

$$\text{But } \frac{\partial f}{\partial x} \Big|_{x, y+\delta y} = \lim_{\delta x \rightarrow 0} \frac{f(x+\delta x, y+\delta y) - f(x, y+\delta y)}{\delta x}$$

$$\text{And } \frac{\partial f}{\partial x} \Big|_{x, y} = \lim_{\delta x \rightarrow 0} \frac{f(x+\delta x, y) - f(x, y)}{\delta x}$$

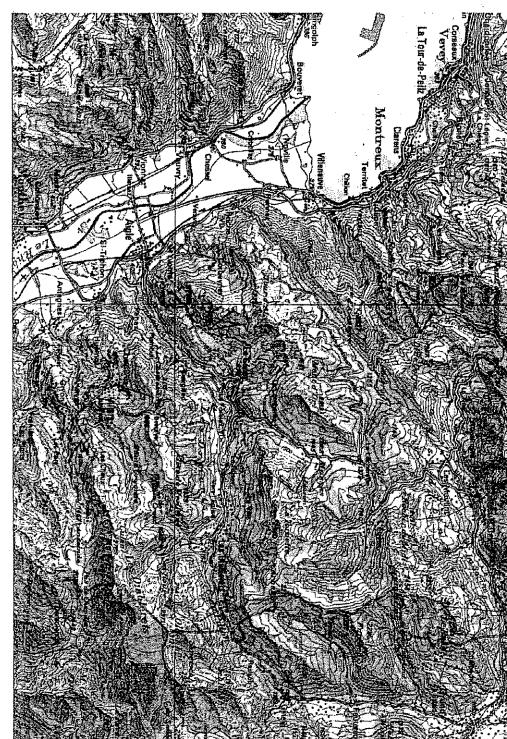
$$\therefore \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) =$$

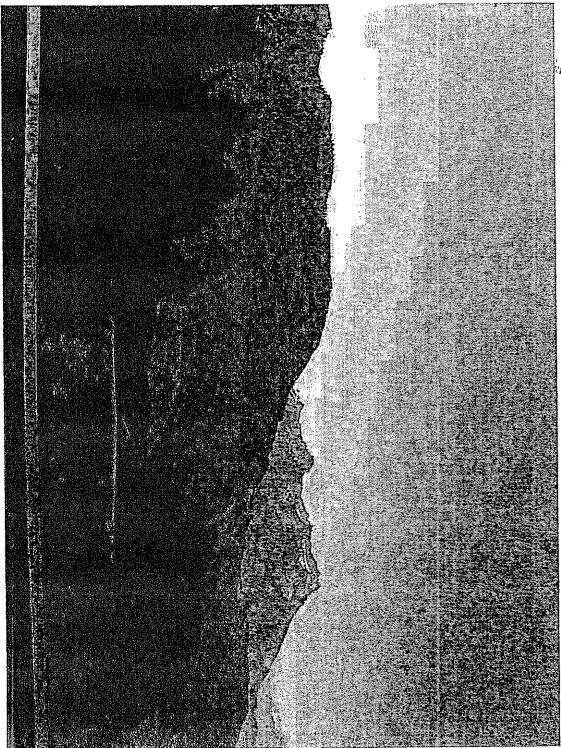
$$\lim_{\delta x \rightarrow 0} \lim_{\delta y \rightarrow 0} \left\{ \left[\frac{f(x+\delta x, y+\delta y) - f(x+\delta x, y)}{\delta y} \right] - \left[\frac{f(x, y+\delta y) - f(x, y)}{\delta y} \right] \right\} \frac{1}{\delta x}$$

$$= \lim_{\delta x \rightarrow 0} \frac{\frac{\partial f}{\partial y} \Big|_{x+\delta x, y} - \frac{\partial f}{\partial y} \Big|_x}{\delta x}$$

$$= \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right)$$

QED.





18.73



18.94

Example

18.10

18.10

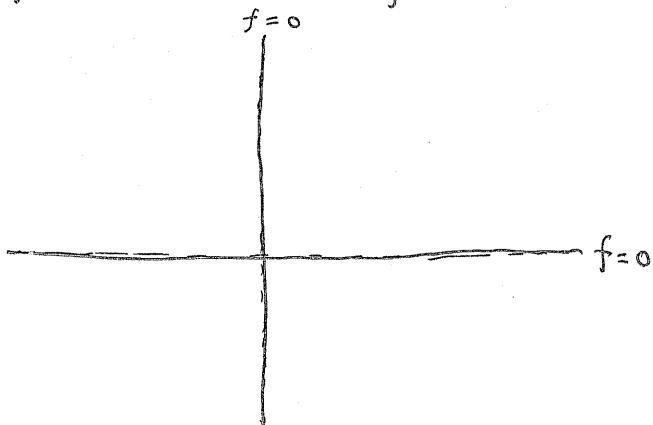
Sketch contour map of

$$f(x, y) = 10 \cos y e^{-0.2(x^2 + y^2)}$$

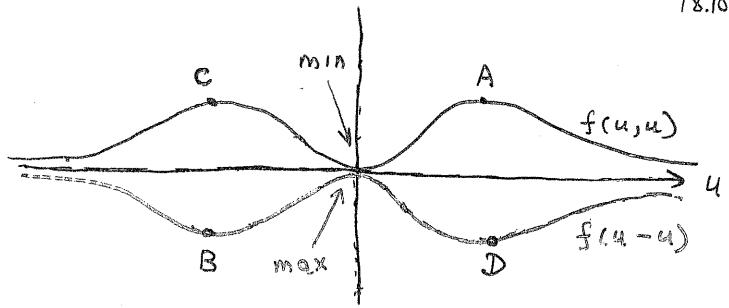
Method

find places where $f(x, y) = 0 \Rightarrow$

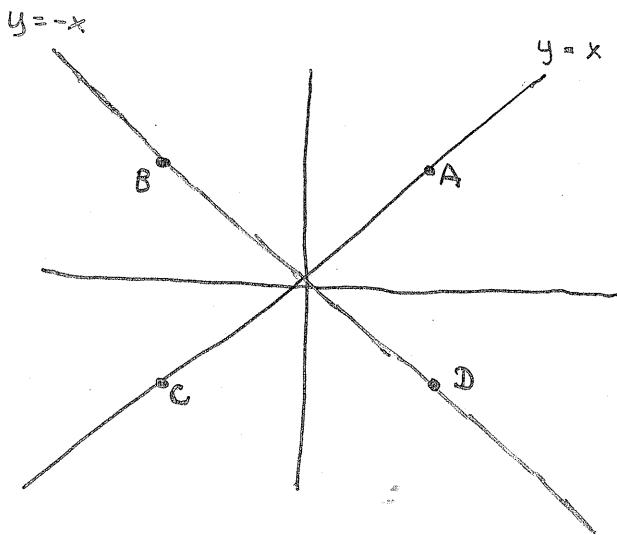
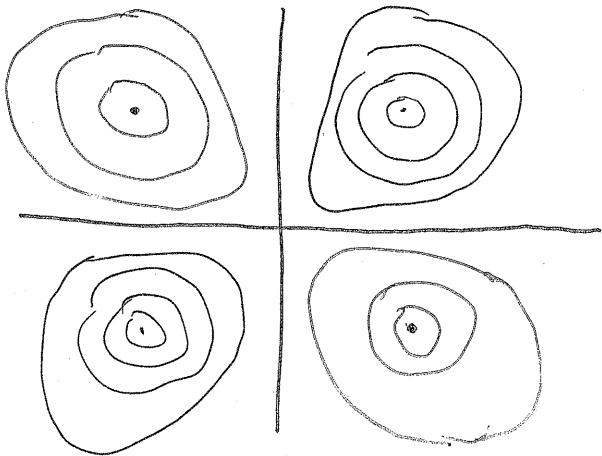
- $f \rightarrow 0$ as $|x|$ or $|y| \rightarrow \infty$
- $f = 0$ if $x = 0$ or $y = 0$


 $f < 0$
 $\therefore \text{min}$
 $f > 0$
 $\therefore \text{max}$
 $f > 0$
 $\therefore \text{max}$
 $f < 0$
 $\therefore \text{min}$

NB for fixed y f behaves like $x e^{-x^2}$
 expect single peak in each quadrant

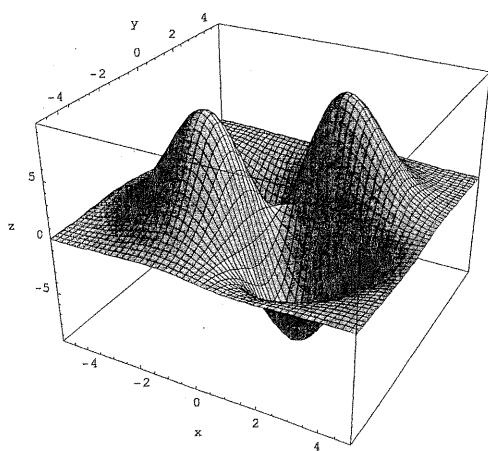
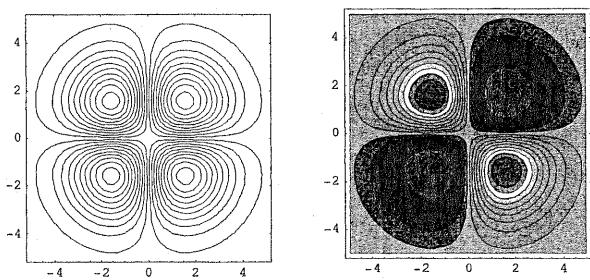


SADDLE POINT AT $(0,0)$



But where exactly are the peaks?
 $(0,0)$ is a saddle point.

Visualisation of 2D Function Contour & Surface Plots



18.4 2D Taylor Theorem

To calculate where the max & min are we need 2D Taylor Thm:

$$\begin{aligned} f(x_0+h, y_0+k) &= f(x_0, y_0) + h \frac{\partial f}{\partial x} \Big|_{x_0, y_0} + k \frac{\partial f}{\partial y} \Big|_{x_0, y_0} \\ &\quad + \frac{1}{2!} \left(h^2 \frac{\partial^2 f}{\partial x^2} \Big|_{x_0, y_0} + 2hk \frac{\partial^2 f}{\partial x \partial y} \Big|_{x_0, y_0} + k^2 \frac{\partial^2 f}{\partial y^2} \Big|_{x_0, y_0} \right) \\ &\quad + \text{terms of Order } h^3, k^3 \end{aligned}$$

where h & k are small displacements from (x_0, y_0)

- Obvious Extension of 1D Theorem
- For $k=0$ get 1D Taylor in x_0+h
- For $h=0$ get 1D Taylor in y_0+k
- The cross term is new

Proof similar to 1D case
- just verify.

Aside: (Not for exam)

An easy way to remember this formula is to notice if we write the operator

$$(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y}) \text{ as } \hat{O}$$

The series is

$$f(x_0+h, y_0+k) = f_{x_0 y_0} + \hat{O} f + \frac{1}{2!} \hat{O}^2 f + \frac{1}{3!} \hat{O}^3 f$$

This also generalises to 3 (or more) variables.

In fact we usually write \hat{O} as

$$\underline{S}_F \cdot \nabla \quad \text{where } \underline{S}_F = (h, k)$$

& the gradient operator grad

$$\text{is } \nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right) \text{ or}$$

$$\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$$

You will see ∇ a lot more next term.

END of Aside.

Thus for a MIN the quantity

$$\left\{ f_{xx} \alpha^2 + 2f_{xy} \alpha + f_{yy} \right\}$$

where $\alpha \equiv \frac{h}{k}$ must be positive

for all values of α (ie all small h, k)

$$\Rightarrow f_{yy} > 0$$

AND NO REAL ROOTS - ie

$$f_{xy}^2 < f_{xx} f_{yy} \quad (\text{implies } f_{xx} \text{ also } > 0)$$

Sim for a MAX need

$$f_{yy} < 0 \quad \& \quad f_{xy}^2 < f_{xx} f_{yy}$$

And for a saddle (implies f_{xx} also < 0)

we MUST HAVE REAL ROOTS

$$\text{ie } f_{xy}^2 > f_{xx} f_{yy}$$

If $f_{xy}^2 = f_{xx} f_{yy}$ we can't decide
- use common sense.

For stationary point the linear 1st order terms in Taylor expansion must vanish, ie

$$\frac{\partial f}{\partial x} = 0 \quad \text{and} \quad \frac{\partial f}{\partial y} = 0 \quad \rightarrow (18.1)$$

This is a necessary condition for a max or min and is the definition of a stationary point.

To Classify a stationary point look at:

$$\begin{aligned} f(x+h, y+k) - f(x, y) &= f_{xx} \frac{h^2}{2} + f_{xy} hk + f_{yy} \frac{k^2}{2} \\ &= \frac{k^2}{2} \left\{ f_{xx} \left(\frac{h}{k}\right)^2 + 2f_{xy} \left(\frac{h}{k}\right) + f_{yy} \right\} \end{aligned}$$

$$\text{where } f_{xx} = \left. \frac{\partial^2 f}{\partial x^2} \right|_{x,y} \text{ etc}$$

and h, k are small enough to be able to neglect higher terms.

Summary

STATIONARY POINT $f_x = 0 \text{ and } f_y = 0$

MAX $f_{xy}^2 < f_{xx} f_{yy} \quad \& \quad f_{yy} < 0$
 $f_{xx} < 0$

MIN $f_{xy}^2 < f_{xx} f_{yy} \quad \& \quad f_{yy} > 0$
 $f_{xx} > 0$

SADDLE $f_{xy}^2 > f_{xx} f_{yy}$

Example $f = 10xy e^{-0.2(x^2+y^2)}$

$$\Rightarrow f_x = (10y - 4x^2y) e^{-0.2(x^2+y^2)}$$

$$\therefore f_x = 0 \Rightarrow y = 0 \quad \text{or} \quad x = \pm \sqrt{\frac{5}{2}}$$

$$\text{Sim } f_y = 0 \Rightarrow x = 0 \quad \text{or} \quad y = \pm \sqrt{\frac{5}{2}}$$

5 Stationary points as expected.

Final Topic. Logistic Map

19.2

"There are more things in heaven and earth ..."

The sequence

$$X_{n+1} = r X_n (1 - X_n)$$

IS DISCRETE EQUIVALENT OF

$$\frac{dy}{dt} = r y \left(1 - \frac{y}{r}\right) \quad \begin{matrix} y \rightarrow X \\ t \rightarrow n \end{matrix}$$

Paper

R May "Simple models with very complex dynamics"

NATURE 261 459 (1976)

Lead to explosion of interest in deterministic chaos in the 1980's.

we are interested in

$$\lim_{n \rightarrow \infty} X_n \rightarrow X$$

For sensible Limit

$$\boxed{1 \leq r \leq 4 \quad 0 \leq X \leq 1}$$

if $r < 1 \quad X_n \rightarrow 0$

if $r > 4 \quad |X_n| \rightarrow \infty$

In range $1 \leq r \leq 4$ limit is

$$X = rX(1-X)$$

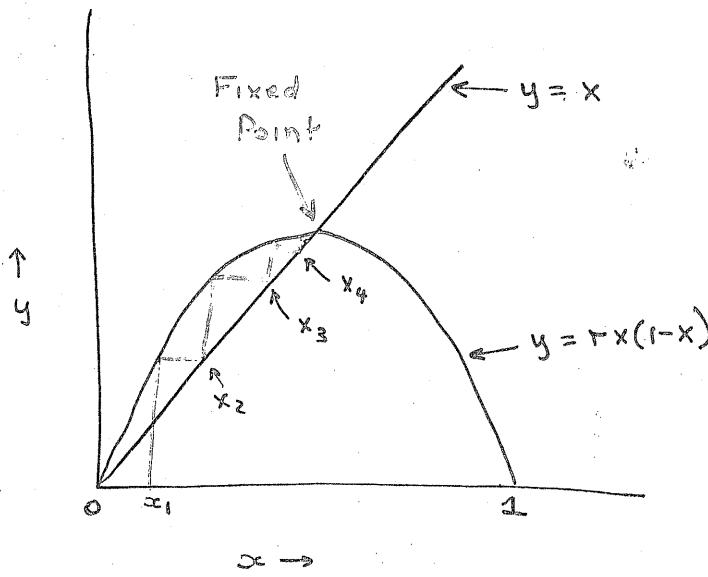
i.e. $X=0$ (not interesting)

or $\boxed{X = 1 - \frac{1}{r}}$

Approach To Limit

19.3

19.

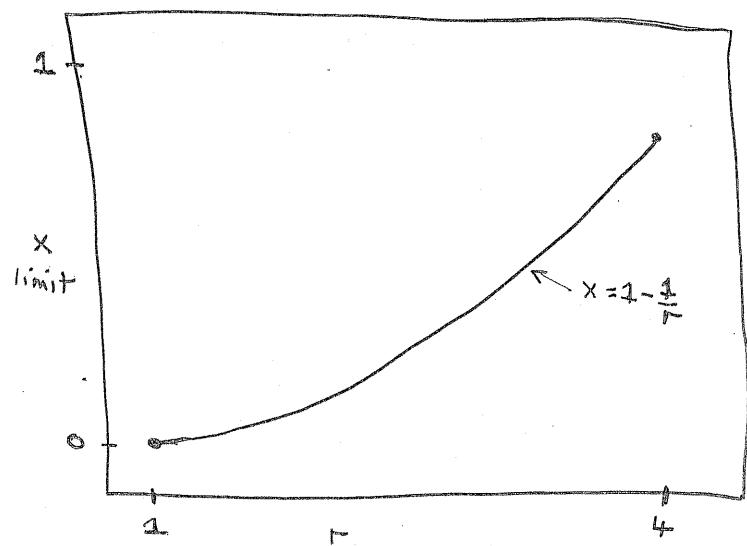


We can perform a numerical experiment

calculate $X_{1000}, X_{1001}, \dots, X_{1400}$

for $X_0 = 0.5$ (could be anything)

For many values of r .



What do we see?