Part IA NT Maths Tripos
Model Answer– Paper II 2008 Y12
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Comments

I was asked to solve this question in the very last supervision I gave May 2012 before retiring at the end of that academic year. Although I thought I had done the question in the past, I was unable to find the solution during that supervision. I did email my brute force solution to the students concerned the next day.

The student who asked for the solution was very capable,she had done all the other questions on several year's worth of past papers as revision but she was stuck on this one!

Find: $I(a,b)=\int_{-\infty}^{\infty} e^{-\frac{1}{2}(ax^2+b/x^2)}dx$ given $\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2}dx=\sqrt{2\pi}$, and the hint, to try the substitution $y=\frac{1}{2}\left(\sqrt{a}x-\sqrt{b}/x\right)$. This part was worth 15 marks and there was a second part to find $J(a,b)=\int_{-\infty}^{\infty}\frac{1}{x^2}e^{-\frac{1}{2}(ax^2+b/x^2)}dx$ for 5 marks.

But a second trick is required to solve this the using the hint! Note that I have ignored the factor of $\frac{1}{2}$ suggested in the hint.

Solution

Try the substitution: $y=\sqrt{a}x-\sqrt{b}/x$.

First note $I(a,b)=2\int_{0}^{\infty} e^{-\frac{1}{2}(ax^2+b/x^2)}dx$ by symmetry; and that *y* goes from $-\infty$ to ∞ as x goes from 0 to ∞. Also

$$y^2=ax^2+b/x^2-2\sqrt{ab} \;\Rightarrow\; ax^2+b/x^2=y^2+2\sqrt{ab}\,.$$

Thus using the substitution we get:

$$I(a,b)=2\int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2-\sqrt{ab}}dx \qquad \text{Eqn 1}$$

And

$$dy=(\sqrt{a}+\sqrt{b}/x^2)\,dx\,. \qquad \text{Eqn 2}$$

so far so good, but dealing with the $\sqrt{b}/x^2$ term in the expression for dx is where people get stuck.

The trick is to spot that the substitution $y=\sqrt{a}x-\sqrt{b}/x$ leads to the quadratic equation $\sqrt{a}x^2-xy-\sqrt{b}=0$ which we can easily solve to get *x* in terms of *y*:

$$x=\frac{y\pm\sqrt{y^2-4\sqrt{ab}}}{2\sqrt{a}} \text{ so } dx=\frac{dy}{2\sqrt{a}}\pm\frac{ydy}{\sqrt{y^2-4\sqrt{ab}}}.$$

Now substituting for dx in Eqn 1:

$$I(a,b)=2e^{-\sqrt{ab}}\left[\int_{-\infty}^{\infty}e^{-\frac{1}{2}y^2}\frac{dy}{2\sqrt{a}}\pm\int_{-\infty}^{\infty}e^{-\frac{1}{2}y^2}\frac{y\,dy}{\sqrt{y^2-4\sqrt{ab}}}\right]$$

This looks horrible but the second term with $y\,dy$ is an odd function so that the second nasty integral must vanish! The first term is just a multiple of the standard Gaussian integral, so:

$$\boxed{I(a,b)=\sqrt{\frac{2\pi}{a}}e^{-\sqrt{ab}}}$$

The second part is much easier, try the (rather obvious) substitution $y=1/x$ in the expression for $I(a,b)$. Noting $dy=-dx/x^2$ so that $-dy/y^2=dx$ we get:

$$I(a,b)=2\int_0^{\infty}e^{-(ax^2+b/x^2)}dx=-2\int_{\infty}^{0}\frac{1}{y^2}e^{-(by^2+a/y^2)}dy$$

$$=+2\int_0^{\infty}\frac{1}{y^2}e^{-(by^2+a/y^2)}dy=J(b,a).$$

Hence $J(a,b)=I(b,a)=\sqrt{\frac{2\pi}{b}}e^{-\sqrt{ab}}$. (Alternatively we can use $J(a,b)=-\frac{1}{2}\frac{\partial I}{\partial b}$.)

Alternative Solution:

Follow Richard Feynman's lead[#] and differentiate $I(a,b)$ w.r.t. the parameter b:

$$\frac{\partial I}{\partial b}=2\int_0^{\infty}-\frac{1}{2x^2}e^{-\frac{1}{2}(ax^2+b/x^2)}dx \qquad \text{Eqn 3}$$

(We note in passing that the RHS is just $-\frac{1}{2}J(a,b)$ which gives us an alternative fast way of doing the second part of the question.)

Using our substitution for y in Eqn 3 we get:

$$\frac{\partial I}{\partial b}=-\int_0^{\infty}\frac{1}{x^2}e^{-\frac{1}{2}y^2-\sqrt{ab}}dx.$$

[#] "Surely You Must Be Joking Mr Feynman", pages 86-87. This book is great fun and a must read for all intending Physicists. Here Feynman states that because he knew how to differentiate integrals w.r.t parameters he could solve problems that other graduate students could not.

Now we can combine expressions to deal with the difficult expression for *dy* in Eqn 2:

$$\sqrt{a}I - 2\sqrt{b}\frac{\partial I}{\partial b} = 2\int_0^\infty e^{-\frac{1}{2}y^2-\sqrt{ab}}(\sqrt{a}+\sqrt{b}/x^2)\,dx,$$
$$= 2e^{-\sqrt{ab}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2}\,dy$$
$$= 2e^{-\sqrt{ab}}\sqrt{2\pi}.$$

Rearranging we get a standard first order differential equation for I:

$$\frac{\partial I}{\partial b} - \frac{1}{2}\sqrt{\frac{a}{b}}I = -\sqrt{\frac{2\pi}{b}}e^{-\sqrt{ab}}$$

We can use an integrating factor $\mu(b) = e^{\int -\frac{1}{2}\sqrt{\frac{a}{b}}db} = e^{-\sqrt{ab}}$ to get:

$$\frac{\partial}{\partial b}(Ie^{-\sqrt{ab}}) = -\sqrt{2\pi}\,\frac{1}{b}e^{-2\sqrt{ab}},$$

And finally $I = \sqrt{\frac{2\pi}{a}}e^{-\sqrt{ab}} + K(a)$ where *K(a)* is an arbitrary function of *a*, but noting the value of the integral for $b = 0$ we set $K = 0$ giving the final answer (for 15 marks):

$$I(a,b) = \sqrt{\frac{2\pi}{a}}e^{-\sqrt{ab}}$$

The last part could be easily done by using the obvious substitution $y = 1/x$ or better by noting we have already found that $J(a,b) = -\frac{1}{2}\frac{\partial I}{\partial b}$.