Discontinuity and Continuity of Definite Properties in the Modal Interpretation.

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abstract Some technical results about discontinuity and continuity of eigenprojections of reduced density operators are discussed in an elementary context. It is argued that these results suggest serious obstacles both to the goal of applying the modal interpretation to measurement theory in the context of the quantum statistical mechanics of macroscopic objects and to the goal of extending the modal interpretation to be compatible with relativistic quantum field theory. The paper is based on joint work with Guido Bacciagaluppi and Pieter Vermaas.

From "The Modal Interpretation of Quantum Mechanics", pages 213–222, edited by D. Dieks and P.E. Vermaas, Kluwer (1998).

1 Technical Results in an Elementary Context.

In this paper, we shall consider the Vermaas-Dieks version of the modal interpretation [1]. Suppose that the Hilbert space $\mathcal{H}$ of the universe takes the form of a tensor product $\mathcal{H} \cong \mathcal{H}_s \otimes \mathcal{H}_e$ where $\mathcal{H}_s$ represents the Hilbert space of states of a system of interest, and $\mathcal{H}_e$ represents the Hilbert space of the environment. Suppose also that the quantum state of the universe is some pure state $|\Psi\rangle\langle\Psi|$. Then the state of system s is the reduced state $(|\Psi\rangle\langle\Psi|)_s$ defined by taking the partial trace of $|\Psi\rangle\langle\Psi|$ over $\mathcal{H}_e$.

$(|\Psi\rangle\langle\Psi|)_s$ is a self-adjoint trace class operator on $\mathcal{H}_s$ and so has a unique spectral resolution of the form

$$(|\Psi\rangle\langle\Psi|)_s = \sum_m p_m P_m \tag{1}$$

where the P_m are orthogonal projections such that $\sum_m P_m = 1$, and the p_m are distinct and $\sum_m p_m \dim P_m = 1$. According to [1], the P_m represent the definite properties of the system s.

$(|\Psi\rangle\langle\Psi|)_s$ also possesses eigendecompositions of the form

$$(|\Psi\rangle\langle\Psi|)_s = \textstyle\sum_n r_n |\psi_n\rangle\langle\psi_n|$$

where $(\psi_n)_n$ is an orthonormal basis for $\mathcal{H}_s$, and $\sum_n r_n = 1$. If all the r_n are distinct, then the eigendecomposition is the spectral resolution, and so is unique – apart from phase factors. More generally, however, any sequence of bases for the subspaces $P_m\mathcal{H}_s$ gives rise to an eigendecomposition. This means that the eigendecomposition is non-trivially non-unique whenever any of these subspaces has dimension greater than unity.

In [2], Bacciagaluppi, Vermaas, and I analyse the properties of the P_m and ψ_n and consider how they change with time, under the assumption of a global Hamiltonian H acting on the total Hilbert space $\mathcal{H}$, so that the reduced density matrix has time dependence

$$\rho(t) = (e^{-itH}|\Psi\rangle\langle\Psi|e^{itH})_s.$$

The first part of this paper reviews results from [2]. The results and examples in this part are quoted from [2] and complete technical details, proofs, and references, may be found there.

An elementary example shows how problems may arise:

example Suppose $\mathcal{H}_s$ is two-dimensional. Consider, for $0 \le \varepsilon \le \frac{1}{2}$, reduced density matrices ρ_ε and σ_ε given by $\rho_\varepsilon = \begin{pmatrix} \frac{1}{2}+\varepsilon & 0 \\ 0 & \frac{1}{2}-\varepsilon \end{pmatrix}$ and $\sigma_\varepsilon = \begin{pmatrix} \frac{1}{2} & \varepsilon \\ \varepsilon & \frac{1}{2} \end{pmatrix}$. As long as $\varepsilon > 0$, ρ_ε and σ_ε each have unique pairs of one-dimensional eigenprojections; $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ for ρ_ε and $\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ and $\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$ for σ_ε. Continuity and stability problems arise because, although these pairs are independent of ε, ρ_ε is arbitrarily close to σ_ε for ε sufficiently small. $\varepsilon = 0$ is the degeneracy point, where $\rho_\varepsilon = \sigma_\varepsilon$, any normalized vector is an eigenvector, and the spectral resolution contains the two-dimensional eigenprojection $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Choosing $\mathcal{H}_s$ to be two-dimensional is sufficient to exhibit most of the technical results from [2]. In this case, the "Bloch sphere" construction allows us to represent the states on $\mathcal{H}_s$ by the points of the unit ball in three-dimensional real space $\mathbb{R}^3$.

The Bloch Sphere.

Let Σ be the set of 2×2 density matrices $\rho = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}$.
ρ is a self-adjoint positive matrix with trace unity, so that

$$\mathrm{tr}(\rho) = \rho_{11} + \rho_{22} = 1, \quad \rho_{12} = \overline{\rho_{21}}, \quad 0 \leq \rho_{11} \leq 1,$$
$$\text{and} \quad 0 \leq \det(\rho) = \rho_{11}\rho_{22} - \rho_{12}\rho_{21} \leq \tfrac{1}{4}.$$

A mapping $\chi : \Sigma \to \mathbb{R}^3$ is defined by

$$\chi(\rho)^1 = \rho_{12} + \rho_{21}, \quad \chi(\rho)^2 = i(\rho_{12} - \rho_{21}), \quad \chi(\rho)^3 = \rho_{11} - \rho_{22}.$$

χ maps Σ into the unit ball $B^3 \subset \mathbb{R}^3$:

$$|\chi(\rho)|^2 = 4\rho_{12}\rho_{21} + \rho_{11}^2 - 2\rho_{11}\rho_{22} + \rho_{22}^2 = (\rho_{11} + \rho_{22})^2 - 4\det\rho = 1 - 4\det\rho \leq 1.$$

χ is a bijection onto B^3 with inverse φ defined by

$$\varphi(\mathbf{x}) = \tfrac{1}{2}\begin{pmatrix} 1 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & 1 - x^3 \end{pmatrix}.$$

$\det\varphi(\mathbf{x}) = \frac{1}{4}(1 - (x^1)^2 - (x^2)^2 - (x^3)^2)$.

ρ is pure $\iff \det\rho = 0 \iff |\chi(\rho)|^2 = 1$. This means that the pure states are mapped onto the surface of the ball.

For any $\mathbf{x}$, $\varphi(\mathbf{x}) + \varphi(-\mathbf{x}) = 1$ so that $\varphi(\mathbf{x})$ and $\varphi(-\mathbf{x})$ commute. In particular, for $|\mathbf{x}|^2 = 1$, $\varphi(\mathbf{x})$ and $\varphi(-\mathbf{x})$ are orthogonal pure states.

χ is an affine isomorphism because, for $0 \leq \lambda \leq 1$,

$$\chi(\lambda\rho + (1 - \lambda)\sigma) = \lambda\chi(\rho) + (1 - \lambda)\chi(\sigma)$$
$$\varphi(\lambda\mathbf{x} + (1 - \lambda)\mathbf{y}) = \lambda\varphi(\mathbf{x}) + (1 - \lambda)\varphi(\mathbf{y}).$$

From this it follows that the state represented by a point $\mathbf{x}$ inside the ball can be decomposed into the orthogonal pure states represented by the end points of the line passing through $\mathbf{x}$ and the centre of the ball. This line is unique unless $\mathbf{x} = \mathbf{0}$. Thus $\mathbf{x}$ is non-degenerate unless $\mathbf{x} = \mathbf{0}$.

As B^3 is a manifold with three (real) dimensions and $\mathbf{0}$ is a manifold of zero dimensions, we have an example of the first of the results which I shall take from [2]:

theorem *The space of degenerate density operators on a finite-dimensional Hilbert space $\mathcal{H}_s$ has 3 dimensions fewer than the space of all density operators on $\mathcal{H}_s$.*

Our discussion in [2] of the continuity of eigenvectors of reduced density matrices is based on work by Rellich on the perturbation theory of linear operators. Rellich's main theorem states, when applied to our case, that if the time-dependence of a density matrix $\rho(t)$ is sufficiently smooth – more precisely, if $\rho(t)$ is an analytic function of t – then it is possible to find eigenvectors of $\rho(t)$ which are themselves analytic functions of t.

By trying to construct a counter-example in the Bloch sphere, it is fairly straightforward to see that some such result must hold in the two-dimensional case. As noted above, the eigenvectors of a non-zero point in B^3 can be found by projecting from the point out to the surface, along the line through the centre. This projection is clearly continuous if we avoid the centre, and, indeed, a continuous choice can be made even if we do go through the centre, unless we "turn a sharp corner" there. Smooth paths do not turn sharp corners.

If the global Hilbert space $\mathcal{H}$ is finite-dimensional, then any Hamiltonian H is bounded, and all reduced states of the form $\rho(t) = (e^{-itH}|\Psi\rangle\langle\Psi|e^{itH})_s$ are analytic functions of t. Rellich's theorem on the existence of analytic eigenvectors can also be applied in the modal interpretation if $\mathcal{H}$ is infinite-dimensional because of the following result:

lemma *If H is a Hamiltonian on a tensor product Hilbert space $\mathcal{H} = \mathcal{H}_s \otimes \mathcal{H}_e$ then there is a dense set of vectors $\Psi \in \mathcal{H}$ such that the reduced density operator $\rho(t) = (e^{-itH}|\Psi\rangle\langle\Psi|e^{itH})_s$ is analytic in t.*

It should be noted that Rellich's result is slightly complicated in the infinite-dimensional case. In finite dimensions, there is a time-dependent basis of analytic eigenvectors for $\mathcal{H}_s$. In infinite dimensions however, even with the best analyticity properties for $\rho(t)$, Rellich's theorem only applies to eigenvectors $\psi(t)$ for which the corresponding eigenvalue $r(t)$ is greater than zero. If $r(t) \to 0$, then it is possible for $\psi(t)$ to disappear.

example *Let $(t_n)_{n\geq 1}$ be any sequence of real numbers (for example, some counting of the rational numbers). Then there is a vector Φ in a Hilbert space $\mathcal{H} = \mathcal{H}_s \otimes \mathcal{H}_e$ and a bounded Hamiltonian H on $\mathcal{H}$ such that the density operator $\rho(t) = (e^{-itH}|\Phi\rangle\langle\Phi|e^{itH})_s$ has an eigenvector disappearing at each point of the sequence $(t_n)_{n\geq 1}$.*

Equating eigenvalue $r(t)$ with probability, suggests that such disappearing eigenvectors may not be a problem of great physical significance. Much more important, in my opinion, is the problem of instability.

We have seen that for a given Hamiltonian H, smooth eigenvectors of the reduced state can be chosen. However, these eigenvectors may change uncontrollably under arbitrarily small changes in H.

This problem also is easily exemplified in the Bloch sphere. Imagine that H depends on a parameter η, and that the reduced state

$$\rho(t, \eta) = (e^{-itH(\eta)}|\Psi\rangle\langle\Psi|e^{itH(\eta)})_s$$

sweeps through the degeneracy point at $t = t_0$ and $\eta = \eta_0$. By considering how the projection from the centre of the Bloch sphere to the surface changes as a reduced state moves close to the centre of the sphere, it is easy to see that, if a suitable choice of parameter dependence can be found, then the eigenvectors of ρ can be made to move, for example, from equator to pole for arbitrarily small change in t. This is the basic idea behind the following example.

example *There exists a Hamiltonian* $H(\eta)$ *on a Hilbert space* $\mathcal{H} = \mathcal{H}_s \otimes \mathcal{H}_e$ *and a vector* $\Psi \in \mathcal{H}$ *such that* $H(\eta)$ *is bounded and depends analytically on the parameter* η *and* $(e^{-itH(\eta)}|\Psi\rangle\langle\Psi|e^{itH(\eta)})_s$ *is jointly analytic in* t *and* η. *However, there exist* t_0 *and* η_0 *such that, for any* $\varepsilon > 0$ *there exist* t_1, t_2, η_1, *and* η_2, *with* $|t_1-t_0|+|t_2-t_0|+|\eta_1-\eta_0|+|\eta_2-\eta_0| < \varepsilon$ *and* $||\xi-\xi'|| > \frac{1}{2}$ *for any pair* (ξ, ξ') *consisting of an eigenvector* ξ *of* $(e^{-it_1H(\eta_1)}|\Psi\rangle\langle\Psi|e^{it_1H(\eta_1)})_s$ *and an eigenvector* ξ' *of* $(e^{-it_2H(\eta_2)}|\Psi\rangle\langle\Psi|e^{it_2H(\eta_2)})_s$.

This suggests that what the modal interpretation takes to be the "real" properties of a physical subsystem may fluctuate uncontrollably under environmental perturbations. A similar problem arises in the very identification of subsystems. According to the modal interpretation, a subsystem is given as a factor space in the Hilbert space of the universe. However arbitrarily small changes in the identification of such factors may give rise to large changes in the properties of the corresponding systems.

A Hilbert space $\mathcal{H}$ of dimension N_sN_e can be expressed as a tensor product $\mathcal{H} = \mathcal{H}_s \otimes \mathcal{H}_e$ by giving an indexed basis $(\chi_{mn})_{m=1}^{N_s}{}_{n=1}^{N_e}$ for $\mathcal{H}$ and equating χ_{mn} with $\varphi_m \otimes \psi_n$ where $(\varphi_m)_{m=1}^{N_s}$ is a basis for $\mathcal{H}_s$ and $(\psi_n)_{n=1}^{N_e}$ is a basis for $\mathcal{H}_e$. Two such expressions corresponding to bases $(\chi_{mn})_{m=1}^{N_s}{}_{n=1}^{N_e}$ and $(\chi'_{mn})_{m=1}^{N_s}{}_{n=1}^{N_e}$ for $\mathcal{H}$ may be considered to be close if the basis vectors χ_{mn} and χ'_{mn} are sufficiently close, for all m and n.

Let $\Psi \in \mathcal{H}$ with $||\Psi|| = 1$, and let $(|\Psi\rangle\langle\Psi|)_s$ (resp. $(|\Psi\rangle\langle\Psi|)_{s'}$) denote the density operator on $\mathcal{H}_s$ defined by

$$\langle\varphi_m|(|\Psi\rangle\langle\Psi|)_s|\varphi_{m'}\rangle = \sum_n \langle\chi_{mn}|\Psi\rangle\langle\Psi|\chi_{m'n}\rangle$$

$$\text{(resp.)} \quad \langle\varphi_m|(|\Psi\rangle\langle\Psi|)_{s'}|\varphi_{m'}\rangle = \sum_n \langle\chi'_{mn}|\Psi\rangle\langle\Psi|\chi'_{m'n}\rangle.$$

If for some $\delta > 0$,

$$\sum_{m=1}^{N_s}\sum_{n=1}^{N_e} ||\chi_{mn} - \chi'_{mn}|| < \delta \tag{2}$$

then, for any $\Psi \in \mathcal{H}$,

$$||(|\Psi\rangle\langle\Psi|)_s - (|\Psi\rangle\langle\Psi|)_{s'}||_1 < 2\delta. \tag{3}$$

example *Choose* $\delta > 0$. *There exists a Hilbert space* $\mathcal{H}$ *which can be expressed as a tensor product* $\mathcal{H} = \mathcal{H}_s \otimes \mathcal{H}_e$ *in two possible ways, corresponding to bases* $(\chi_{mn})_{m=1}^{N_s}{}_{n=1}^{N_e}$ *and* $(\chi'_{mn})_{m=1}^{N_s}{}_{n=1}^{N_e}$ *which are close by criteria (2) and (3). There is a vector* $\Psi \in \mathcal{H}$ *such that* $||\xi - \xi'|| > \frac{1}{2}$ *for any pair* (ξ, ξ') *consisting of an eigenvector* ξ *of* $(|\Psi\rangle\langle\Psi|)_s$ *and an eigenvector* ξ' *of* $(|\Psi\rangle\langle\Psi|)_{s'}$.

2 The problem of instability.

In my opinion, instability near degeneracy points is a fundamental problem for the modal interpretation. Theoretical physics is based on a long chain of approximations.

Instability means that eigendecompositions do not behave well under approximation. This makes it impossible for the modal interpretation to claim that it is dealing with any sort of approximation to the truth, while it continues to rely on non-relativistic and few-dimensional models of quantum mechanics. Indeed, near-degeneracies and degeneracies are inevitable for real physical systems according both to the many-dimensional models suggested by quantum statistical mechanics and to the infinite-dimensional models of local relativistic quantum field theory.

Macroscopic systems have high entropy and, therefore, according to quantum statistical mechanics, they should be assigned highly mixed, nearly degenerate states with correspondingly uncontrollable eigenfunctions. These states appear to be quantum mechanical analogs of the ensembles of classical statistical mechanics. However, this does not mean that the interpretation of these states is entirely straightforward. Indeed, the modal interpretation in general can be seen as an attempt to grapple with the problem of extending an ensemble picture to quantum states by providing an exact and unambiguous definition of the ensemble which corresponds to such a state. This problem is just the same for states of macroscopic quantum systems as it is for microscopic quantum systems.

There is also a question as to whether the high entropy states of quantum statistical mechanics are the states which we should assume that the corresponding systems occupy. Consider, for example, a hot cup of coffee. According to a direct application of the modal interpretation to macroscopic objects, this coffee has a quantum state $(|\Psi\rangle\langle\Psi|)_c$, which is the reduction of the state of the entire universe to the Hilbert space $\mathcal{H}_c$ defined by the particles making up the coffee. More precisely, $(|\Psi\rangle\langle\Psi|)_c$ is the reduction to $\mathcal{H}_c$ of the state of the entire universe, given a considerable amount of prior information; for example, given that the cup of coffee exists, or given the observations of the coffee drinker. Statistical mechanical arguments then suggest that the most plausible guess we can make for $(|\Psi\rangle\langle\Psi|)_c$ is that it is the state on $\mathcal{H}_c$ with highest entropy given our prior information. If the modal interpretation is really the complete and universal interpretation of quantum mechanics which it purports to be, then appropriate prior information would correspond to fixed definite properties on some suitable systems. If we assume that the prior information is limited by our observations of the coffee, then $(|\Psi\rangle\langle\Psi|)_c$ will be a high-entropy, near-degenerate, quasi-equilibrium, thermal state.

Absolute entropies S at 25°C are 2.4 J K^{-1} mol^{-1} for diamond and 205 J K^{-1} mol^{-1} for oxygen. If we equate S with $k_B \log N$, then N is a measure of the minimum number of orthogonal wave-functions into which the equilibrium quantum state decomposes with significant probability. $N \sim 10^{6.3\times10^{21}}$ for one gram of diamond and $N \sim 10^{2.6\times10^{23}}$ for one litre of oxygen.

Also relevant in this context, is work by Lubkin [3] and Page [4], which has recently been turned into a theorem by Foong and Kanno [5]. These authors have shown that most pure states on a Hilbert space of sufficiently large dimension give rise to nearly maximally degenerate states on restriction to a subspace of appropriate dimension. More precisely, they have shown that if $|\Psi\rangle\langle\Psi|$ is a randomly chosen pure state on a Hilbert space $\mathcal{H} = \mathcal{H}_s \otimes \mathcal{H}_e$ of dimension $N_s N_e$ with $N_e \gg N_s \gg 1$ then

$$S((|\Psi\rangle\langle\Psi|)_s) = \mathrm{tr}(-(|\Psi\rangle\langle\Psi|)_s \log(|\Psi\rangle\langle\Psi|)_s) \sim \log N_s.$$

In the Vermaas-Dieks form of the modal interpretation, the possessed properties are fixed by an algorithm; the possessed properties correspond to the eigenprojections of the density matrix of the subsystem. Having such an algorithm is an enormous advance compared to the conventional interpretation of quantum theory, in which we are supposed to look for the eigenfunctions of some "measured" operator, but no precise means is provided by which we can identify that operator. Nevertheless, in my opinion, the motivation behind the modal interpretation remains the suggestion that the mysterious nature of quantum mechanical states can be resolved if a subsystem "possesses" properties, which are (somehow) "quasi-classical". Thus, in the modal interpretation also, it is implied that the possessed properties should, in some sense, correspond to definite values of what is being measured. Unfortunately, the algorithm supplied by the Vermaas-Dieks modal interpretation does not necessarily yield "quasi-classical" properties. The eigenfunctions of a reduced density matrix near an N-fold degeneracy vary over an N-dimensional space.

It has been suggested that decoherence theory solves this problem. This suggestion is incorrect. Decoherence theory does tell us that the reduced state is close to a state which has a decomposition into pure states with physically desirable properties, but this is NOT equivalent to saying that the eigendecomposition of the reduced state is into pure states with close to physically desirable properties. Once again, the algorithmic nature of the modal interpretation, which is its greatest strength, makes it impossible for the interpretation to hide behind the usual "for all practical purposes" (FAPP) arguments. The following examples demonstrate this point. Similar examples will be presented by Bacciagaluppi [6] in a forthcoming paper.

example The one-particle reduced density matrix for a one-dimensional ideal gas of particles of mass m confined to an interval $[0, L]$ in the classical regime at temperature T is given by

$$\rho_{\beta,L}(x,y) = \frac{1}{Z} e^{-\beta H}(x,y) = \frac{2}{ZL} \sum_{n=1}^{\infty} e^{-\alpha n^2} \sin \frac{n\pi x}{L} \sin \frac{n\pi y}{L}$$

$$\text{where } \beta = 1/kT, \alpha = \frac{\hbar^2 \pi^2}{2mL^2 kT}, \text{ and } Z = \mathrm{tr}(e^{-\beta H}) = \sum_{n=1}^{\infty} e^{-\alpha n^2}.$$

The eigenfunctions of $\rho_{\beta,L}$ given by $\sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L}$ are unique and non-localized, but, by expanding $\frac{1}{2}\sqrt{\frac{\pi}{\alpha}} \sum_{s=-\infty}^{\infty} e^{-\pi^2 (x+2s)^2/4\alpha}$ in cosines, it is possible to show that, for $|x-y| \ll x, y \ll L$,

$$\rho_{\beta,L}(x,y) \sim \frac{1}{L} e^{-\frac{m}{2\hbar^2 \beta}(x-y)^2}.$$

Thus $\rho_{\beta,L}$ has a decoherence length $\sqrt{\frac{2\hbar^2\beta}{m}}$, corresponding to the de Broglie thermal wavelength, which is of order 10^{-11} m for atoms at room temperature. This means that $\rho_{\beta,L}$ is exactly the type of density operator which decoherence theory claims is typical for a macroscopic system. Nevertheless, the eigenfunctions of ρ are utterly "quantum mechanical" in nature.

example Consider a particle of mass m in a harmonic potential in one dimension. The Schrödinger equation is:

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + \frac{1}{2}m\omega^2x^2\Psi.$$

At temperature T, the density matrix of the particle is

$$\rho_\beta = (1 - e^{-\beta\hbar\omega})\sum_{n=0}^{\infty} e^{-\beta n\hbar\omega}|\psi_n\rangle\langle\psi_n|$$

where $\beta = 1/kT$.

The harmonic oscillator eigenfunctions ψ_n are independent of temperature and have length scale $\sqrt{\frac{(n+\frac{1}{2})\hbar}{m\omega}}$. Again, these are utterly "quantum mechanical" eigenfunctions.

Using the generating function for the ψ_n, it can be shown that

$$\rho_\beta(x,y) = \sqrt{\frac{m\omega}{\pi\hbar\lambda}}e^{-\frac{\lambda m\omega}{4\hbar}(x-y)^2 - \frac{m\omega}{4\hbar\lambda}(x+y)^2}$$

where $\lambda = \frac{1+e^{-\beta\hbar\omega}}{1-e^{-\beta\hbar\omega}}$.

In the high temperature limit, $\beta \to 0$, and $\lambda \sim 2/(\beta\hbar\omega)$, so that

$$\rho_\beta(x,y) \sim \sqrt{\frac{m\omega^2\beta}{2\pi}}e^{-\frac{m}{2\hbar^2\beta}(x-y)^2 - \frac{m\omega^2\beta}{8}(x+y)^2}.$$

This has the same form as the exact result. The correlation length at high temperature $\sqrt{\frac{2\hbar^2\beta}{m}}$ is the same as that for the free particle. The term $e^{-\frac{m\omega^2\beta}{8}(x+y)^2}$ provides a long length-scale decrease, so that ρ_β is normalized. Once again, this is a "decoherent" density operator.

Other decompositions of ρ_β are possible. For example, following Glauber [7], but using an explicit width parameter ξ, ρ_β can also be represented as a Gaussian distribution of coherent states

$$\psi_{u,v,\xi}(x) = \frac{1}{\sqrt{\sqrt{\pi}\xi}}e^{-(x-u)^2/2\xi^2 + ivx/\hbar}.$$

For $\frac{\hbar\lambda}{m\omega} > \xi^2 > \frac{\hbar}{\lambda m\omega}$,

$$\rho_\beta = \sqrt{\frac{m\omega\xi^2}{\pi^2(\hbar\lambda-\xi^2 m\omega)(\lambda m\omega\hbar\xi^2-\hbar^2)}} \int |\psi_{u,v,\xi}\rangle\langle\psi_{u,v,\xi}|e^{-\frac{m\omega}{\hbar\lambda-\xi^2 m\omega}u^2 - \frac{\xi^2}{\lambda m\omega\hbar\xi^2-\hbar^2}v^2}dudv.$$

These decompositions decompose ρ_β into a range of well-localized quasi-classical particle states. Such states are satisfactory "for all practical purposes". It would be splendid if an algorithmic interpretation could be used to break the non-uniqueness represented by ξ and pick out exactly one of these decompositions. The modal interpretation algorithm, of course, does not do this.

The problems raised for the modal interpretation by quantum statistical mechanics are serious. At the very least, instability near degeneracy points implies that the analysis of prior information in the modal interpretation is not a problem which can be ignored; the modal interpretation cannot take advantage of anything analogous to the free choice of Heisenberg cut between measuring apparatus and measured system. However, the problems raised by relativistic quantum field theory are perhaps even more fundamental. The modal interpretation is supposed to be a universal "no collapse" theory. This means that our ultimate goal should be to analyse a universal wave-function $\Psi \in \mathcal{H}$ which would be an uncollapsed state arising from the big bang. Ψ would be a superposition of all possibilities. In any regime of space-time, Ψ would be close to a thermal equilibrium state. Even stars would be superposed in Ψ. Until we began the process of assigning definite properties, there would be no definite macroscopic objects; no measuring devices in ready states. In this context, it seems to me that the only natural subsystems with which we can start our analysis of the universal wave-function, are the "local algebras" of the Haag-Schroer-Kastler axioms [8]. However, these local algebras are type III von Neumann algebras and, as such, they have *no* pure normal states. It is possible to define reduced states on such algebras, but these states have *no* eigendecompositions which correspond in any relevant way to analogs of (1). In a very real sense, in relativistic quantum field theory, local systems are irreducibly degenerate.

3 Where does the modal interpretation go from here?

One possibility for modifying the Vermaas-Dieks modal interpretation might be to consider an alternative algorithm yielding alternative decompositions. Some sort of maximum entropy decomposition might well be desirable, but there would be problems with continuous distributions, and with infinite dimensional systems.

A second possibility would be to use decoherence theory to say that there are suitable decompositions. This, of course, is mere FAPP.

The path I favour, involves going back to Everett, who was the first to use the Schmidt decomposition as the technical foundation of an interpretation, and starting again. My own version of the many-minds interpretation [9] is also an algorithmic interpretation. However, unlike the modal interpretation, it is fully compatible with relativistic quantum field theory and it is mathematically stable under approximation. Working in the context of a universal "no collapse" theory, it involves a detailed, mathematical, analysis of the structure of observers, which takes into account the macroscopic, localized, and thermal nature of observers. It does not attempt to associate an individual wavefunction at each moment to an observer. In my opinion, all such attempts are essentially unphysical, ignoring as they do, not only the mathematics of relativity, but also the continuous and unavoidable interactions between a warm, breathing observer and his environment. Instead, in my theory, observers are taken to occupy suitable mixed thermal states.

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