Continuity and Discontinuity of Definite Properties in the Modal Interpretation.

By Guido Bacciagaluppi,

Faculty of Philosophy, University of Cambridge, Sidgwick Avenue,
Cambridge CB3 9DA, UK E-mail: gb120@ cus.cam.ac.uk

Matthew J. Donald,

The Cavendish Laboratory, University of Cambridge, Madingley Road,
Cambridge CB3 0HE, UK E-mail: mjd1014@ phy.cam.ac.uk
(address for correspondence)

and Pieter E. Vermaas

Foundations of the Natural Sciences Section, Utrecht University,
P.O. Box 80.000, 3508 TA Utrecht, The Netherlands E-mail: p.e.vermaas@ fys.ruu.nl

Abstract Technical results about the time dependence of eigenvectors of reduced density operators are considered, and the relevance of these results is discussed for modal interpretations of quantum mechanics which take the corresponding eigenprojections to represent definite properties. Continuous eigenvectors can be found if degeneracies are avoided. We show that, in finite dimensions, the space of degenerate operators has co-dimension 3 in the space of all reduced operators, suggesting that continuous eigenvectors almost surely exist. In any dimension, even when degeneracies are hit, we find conditions under which a theorem due to Rellich can provide continuous eigenvectors. We use this result to formulate an extended version of the modal interpretation. We also discuss eigenvector instability which we argue poses a serious problem for the modal interpretation, even in our extended version. Many examples are given to illustrate the mathematics.

Helvetica Physica Acta, Volume 68, pages 679–704 (1995).

1 Introduction.

In this paper, we shall consider some technical results about the time-dependent behaviour of the eigenvectors, or eigenprojections, of reduced density operators and discuss the relevance of these results for modal interpretations of quantum mechanics. The idea common to the modal interpretations of quantum mechanics, in the versions by Kochen [1], Krips [2], Dieks [3], Healey [4], Vermaas and Dieks [5], and Clifton [6], is that, by using the spectral resolution of a system's reduced density operator, one can always attribute certain (generally time-dependent) properties to a quantum system. We shall focus on the version of the modal interpretation presented in [5]. A brief survey of other versions is given in [7].

Suppose a quantum mechanical system S to be defined on a Hilbert space $\mathcal{H}_s$, with its environment defined on $\mathcal{H}_e$ and the total Hilbert space of the universe taking the form $\mathcal{H} = \mathcal{H}_s \otimes \mathcal{H}_e$. Given a density operator state on $\mathcal{H}$, S can be assigned a *reduced density operator* ρ, by partially tracing over the degrees of freedom of $\mathcal{H}_e$. ρ will also be referred to as the *reduced state*. We shall assume, in general, that the state on $\mathcal{H}$ is a pure state $|\Psi\rangle\langle\Psi|$, in which case ρ, which will be denoted by $(|\Psi\rangle\langle\Psi|)_s$, is closely related to the Schmidt decomposition of the vector Ψ. This means that our results are also relevant to naive versions of the many-worlds interpretation. Indeed, confrontation with some of the same problems that we shall raise here for modal interpretations was a motivation for one of us to develop a technically sophisticated version of the many-worlds interpretation [8].

Whether or not the state on the total Hilbert space is pure, ρ may well be mixed. ρ has a unique spectral resolution of the form

$$\rho = \sum_m p_m P_m, \tag{1.1}$$

with $0 \leq p_m \leq 1$, $\sum_m p_m = 1$, and $p_m \neq p_n$ whenever $m \neq n$. As density operators are compact operators, this spectral resolution is discrete and $p_m > 0$ implies that P_m is a finite-dimensional projection. Vermaas and Dieks [5] interpret the projections P_m in the spectral resolution of the reduced state as representing definite properties of the system, corresponding to propositions which are either true or false. For each m, $p_m \dim(P_m)$ is the probability that P_m is the property that is actually possessed and corresponds to a true proposition, and the P_n with $n \neq m$ are then actually not possessed and correspond to false propositions.

ρ also has "eigenvector decompositions" which take the form

$$\rho = \sum_{n=1}^{N} r_n |\psi_n\rangle\langle\psi_n|, \tag{1.2}$$

where N denotes the dimension of $\mathcal{H}_s$ (N may or may not be finite), $(\psi_n)_{n=1}^N$ is an orthonormal basis of $\mathcal{H}_s$, and $(r_n)_{n=1}^N$ is a sequence of non-negative real numbers summing to one. The sequence $(r_n)_{n=1}^N$ is unique if the eigenexpansion is "ordered" in the sense that $r_1 \geq r_2 \geq \ldots$. The sequence of the p_m of (1.1) is the sequence of distinct values of the r_n, and the eigenprojections P_m of (1.1) are given by $P_m = \sum\{|\psi_n\rangle\langle\psi_n| : r_n = p_m\}$. The ψ_n however are not unique; any union of orthonormal bases of the subspaces $P_m\mathcal{H}_s$ will give a possible sequence of eigenvectors. The non-uniqueness is insignificant only if all the $P_m\mathcal{H}_s$ are one-dimensional, in which case it corresponds merely to the possibility of multiplication of each ψ_n by an arbitrary phase factor.

This paper addresses questions connected with the continuity and stability of the time evolution of these decompositions of reduced states. In general, we shall consider time evolution driven by a Hamiltonian acting on the total Hilbert space $\mathcal{H}$. We shall see that not only can the continuity of the eigenvalues r_n be established, but also that it is possible, under fairly mild conditions (laid out in section 4), to find eigenvectors which are analytic in time; even at instants when the dimensions of the P_m change (degeneracy points). In addition to these questions of continuity, we shall also examine the quite separate question of the stability of eigenprojections in the neighbourhood of a degeneracy.

example 1.3 Suppose that $\mathcal{H}_s$ is two-dimensional. Consider, for $0 \leq \varepsilon \leq \frac{1}{2}$, reduced density matrices ρ_ε and σ_ε given in some fixed basis by $\rho_\varepsilon = \begin{pmatrix} \frac{1}{2}+\varepsilon & 0 \\ 0 & \frac{1}{2}-\varepsilon \end{pmatrix}$ and $\sigma_\varepsilon = \begin{pmatrix} \frac{1}{2} & \varepsilon \\ \varepsilon & \frac{1}{2} \end{pmatrix}$. Then, as long as $\varepsilon > 0$, ρ_ε and σ_ε each have unique pairs of one-dimensional eigenprojections, given by $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ for ρ_ε and by $\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ and $\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$ for σ_ε. Continuity and stability problems arise because, although these pairs are independent of ε, ρ_ε is arbitrarily close to σ_ε for ε sufficiently small. $\varepsilon = 0$ is the degeneracy point, where $\rho_\varepsilon = \sigma_\varepsilon$, any normalized vector is an eigenvector, and the spectral resolution (1.1) contains only the two-dimensional eigenprojection $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

example 1.4 Ignoring for the moment the question of existence of total spaces and Hamiltonians, possible time-dependent two-dimensional density operators include

A) $\rho(t) = \begin{pmatrix} \frac{1}{2}+t & 0 \\ 0 & \frac{1}{2}-t \end{pmatrix}$ for $-\frac{1}{2} \leq t \leq \frac{1}{2}$. In this case, the degeneracy point is passed through at $t = 0$, but a continuous eigenvector decomposition is given by

$$\rho(t) = (\tfrac{1}{2}+t)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + (\tfrac{1}{2}-t)\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

B) $\rho(t) = \begin{pmatrix} \frac{1}{2}+t & 0 \\ 0 & \frac{1}{2}-t \end{pmatrix}$ for $-\frac{1}{2} \leq t \leq 0$, $\rho(t) = \begin{pmatrix} \frac{1}{2} & t \\ t & \frac{1}{2} \end{pmatrix}$ for $0 \leq t \leq \frac{1}{2}$. In this case, $\rho(t)$ is continuous, but there are no continuous eigenvectors.

In the next three sections, we discuss general mathematical results relevant to problems of continuity of eigenprojections. In section 3, we argue that it is possible to claim, at least for finite-dimensional systems, that degeneracy points are non-generic and so have zero probability of ever being hit by real physical systems. In section 4, we present the most important and surprising of the continuity results. This is a consequence of a theorem due to Rellich. It shows that, under fairly mild conditions on Ψ and leaving aside eigenprojections with vanishing eigenvalue, a reduced density operator has an eigendecomposition into eigenprojections which, in some neighbourhood, evolve analytically, even where there are degeneracy points. Section 5 is devoted to examples relevant to this result, demonstrating the behaviour which is possible for eigenprojections with vanishing eigenvalue and showing that discontinuous evolution of eigenprojections is possible if our conditions on Ψ are not satisfied. In section 6, we consider the implications of eigenprojection analyticity for the modal interpretation. In this case, one can define an eigenvector decomposition of ρ that is continuous even at degeneracy points. This allows us to define "assignable" projections that are finer than the eigenprojections in the spectral resolution (1.1), and we suggest using these to attribute properties to S. This extension of the modal interpretation does not suffer from the discontinuity problems of the version based only on the spectral resolution (1.1).

This investigation is obviously relevant to the problem of defining dynamics for actually possessed properties in the modal interpretation. Nevertheless, it should be noted that, because of the time dependence of the probabilities imposed by the modal interpretation, the continuous evolution of the assignable projections cannot, in general, represent the only way in which possessed properties may change in time; the property possessed by an individual system must also make random jumps from one continuously changing property to another.

Instabilities in the neighbourhood of a degeneracy are not ruled out by these continuity results. In section 7, we shall demonstrate that arbitrarily small variations, either in the pure state $|\Psi\rangle\langle\Psi|$ on $\mathcal{H}$, or in the splitting of $\mathcal{H}$ by which the system S is defined, can induce significant variations in the eigenprojections of the reduced state. Questions linked to such instabilities have not been raised explicitly in the modal interpretation. However, the possible sensitivity of eigenprojections to the exact form of the state has been used by Albert and Loewer [9, 10] in their criticism of the modal interpretation's account of measurements. Their work is discussed extensively in [7]. We shall argue that, quite in general, these instabilities pose a serious problem for the modal interpretation, even in our extended version.

The paper concludes with a brief discussion and with some further examples.

2 Continuity.

Questions about the continuity of the eigenvalues, eigenvectors and eigenprojections of an operator depending on a parameter have been intensively studied for many years. Much original work on the mathematical theory was done by Rellich (see his lectures, [11]).

A development and exposition of his work is given in the textbook by Kato [12] and a brief survey in [13], chapter XII. In this section and in section 4, we shall review some of these results in the comparatively simple context in which we wish to apply them. Background for the mathematics used in these sections is widely available. In particular, we would mention [14], chapter VI, for the theory of trace class operators and [15], chapter IX, for the theory of analytic functions with values in a Banach space.

lemma 2.1 *Let ρ, σ be density operators on a Hilbert space $\mathcal{H}_s$ and let $(r_n(\rho))_{n=1}^N$, $(r_n(\sigma))_{n=1}^N$ be the corresponding sequences of ordered eigenvalues, as in equation (1.2). Then $|r_n(\rho) - r_n(\sigma)| \leq ||\rho - \sigma||_1$ for $n = 1, \dots, N$.*

proof (Amplifying a remark by Simon [16]).

The min-max theorem states that

$$r_n(\rho) = \max\{\min\{(\psi, \rho\psi) : ||\psi|| = 1, \psi \in V\} : V \text{ is an } n\text{-dimensional subspace of } \mathcal{H}_s\}.$$

Choose for V the space spanned by eigenvectors $(\varphi_k)_{k=1}^n$ of σ with $\sigma\varphi_k = r_k(\sigma)\varphi_k$. Let $\varepsilon = ||\rho - \sigma||_1$. Then

$$\begin{aligned} r_n(\rho) &\geq \min\{(\psi, \rho\psi) : ||\psi|| = 1, \psi \in V\} \\ &\geq \min\{(\psi, \sigma\psi) - \varepsilon : ||\psi|| = 1, \psi \in V\} = (\varphi_n, \sigma\varphi_n) - \varepsilon = r_n(\sigma) - \varepsilon. \end{aligned}$$

Exchanging ρ and σ, the result follows. ∎

In the statement of this result, we have used the physically relevant norm for density operators which is the trace norm, denoted by $||\ ||_1$. The proof, however, is valid also for the operator norm. It is an immediate consequence of this lemma that if $\rho(t)$ is a continuous density-operator-valued function of t then, for each n, the n^{th} ordered eigenvalue $r_n(\rho(t))$ is a continuous function of t.

For $\Delta \subset \mathbb{R}$, let $P_\Delta(\rho)$ denote the spectral projection of ρ, so that, with the notation introduced above,

$$P_\Delta(\rho) = \sum\{|\psi_n(\rho)\rangle\langle\psi_n(\rho)| : r_n(\rho) \in \Delta\}.$$

Suppose that $a < b$ and that a and b are not eigenvalues of ρ. If $\dim \mathcal{H}_s = \infty$ then suppose also that $a, b \neq 0$. If σ_k converges in norm to ρ, then $P_{(a,b)}(\sigma_k)$ converges in norm to $P_{(a,b)}(\rho)$. This is a standard result that can be proved using the convergence of the resolvents. Indeed, a similar result holds for general bounded self-adjoint operators ([14], theorem VIII.23(b)). We shall give an alternative proof of a somewhat stronger result for the situation of present interest, that provides explicit error bounds.

lemma 2.2 *Let ρ be a density operator on $\mathcal{H}_s$. Suppose that $a < b$ and that a and b are not eigenvalues of ρ. If $\dim \mathcal{H}_s = \infty$ then suppose also that $a, b \neq 0$. Choose $\varepsilon \in (0, \frac{1}{2})$ such that*

$$\inf\{|r_n(\rho) - a|\} > \varepsilon, \quad \textit{and} \quad \inf\{|r_n(\rho) - b|\} > \varepsilon.$$

Then, for any density operator σ such that $||\sigma - \rho||_1 < \frac{4}{9}\varepsilon^2$,

$$\mathrm{tr}((P_{(a,b)}(\sigma) - P_{(a,b)}(\rho))^2) < \varepsilon.$$

proof Suppose that the conditions of the lemma hold. Write $P_1 = P_{(a,b)}(\rho)$ and $P_2 = P_{(a,b)}(\sigma)$.

By lemma 2.1, $\inf\{|r_n(\sigma) - a|\} > \varepsilon - \frac{4}{9}\varepsilon^2 > \frac{7}{9}\varepsilon$ and $\inf\{|r_n(\sigma) - b|\} > \varepsilon - \frac{4}{9}\varepsilon^2 > \frac{7}{9}\varepsilon$, so that $\mathrm{tr}(P_1)$ and $\mathrm{tr}(P_2)$ are both equal to the number of elements of $\{r_n(\rho) : a < r_n(\rho) < b\}$.

We may assume without loss of generality that $\mathrm{tr}(P_1) < \infty$, as $\mathrm{tr}(P_1) = \infty$ only if $\dim \mathcal{H}_s = \infty$ and $a < 0$. In this case, either the result is obvious (for $b < 0$ or $b \geq 1$), or $a < 0 < b < 1$. But then $1 - P_1 = P_{(b,1]}(\rho)$, $1 - P_2 = P_{(b,1]}(\sigma)$, $\mathrm{tr}(1 - P_1) < \infty$, $\mathrm{tr}((P_{(a,b)}(\sigma) - P_{(a,b)}(\rho))^2) = \mathrm{tr}(((1 - P_1) - (1 - P_2))^2)$ and the result follows from the result for the interval $(b, 1]$.

$$\text{Write} \quad P_3 = P_{(b,1]}(\sigma) \quad \text{if} \;\; b < 1 \quad \text{and} \quad P_3 = 0 \quad \text{if} \;\; b \geq 1.$$

$$\text{Write} \quad P_4 = P_{[0,a)}(\sigma) \quad \text{if} \;\; a > 0 \quad \text{and} \quad P_4 = 0 \quad \text{if} \;\; a \leq 0.$$

$P_2 = 1 - P_3 - P_4$ as, by lemma 2.1, $P_{\{a\}}(\sigma) = P_{\{b\}}(\sigma) = 0$.

$$\begin{aligned}\mathrm{tr}(\rho P_1 P_3) &= \sum \{r_n(\rho) <\psi_n(\rho)|P_3|\psi_n(\rho)> : a < r_n(\rho) < b\} \\ &\leq (b - \varepsilon)\,\mathrm{tr}(P_1 P_3).\end{aligned}$$

Similarly, $\mathrm{tr}(\sigma P_1 P_3) \geq (b + \frac{7}{9}\varepsilon)\,\mathrm{tr}(P_1 P_3)$, $\mathrm{tr}(\rho P_1 P_4) \geq (a + \varepsilon)\,\mathrm{tr}(P_1 P_4)$, and $\mathrm{tr}(\sigma P_1 P_4) \leq (a - \frac{7}{9}\varepsilon)\,\mathrm{tr}(P_1 P_4)$.

Thus, $\frac{4}{9}\varepsilon^2 > \mathrm{tr}((\sigma - \rho) P_1 P_3) \geq \frac{16}{9}\varepsilon\,\mathrm{tr}(P_1 P_3)$, $\frac{4}{9}\varepsilon^2 > \mathrm{tr}((\rho - \sigma) P_1 P_4) \geq \frac{16}{9}\varepsilon\,\mathrm{tr}(P_1 P_4)$,

$$\text{and} \quad \mathrm{tr}((P_1 - P_2)^2) = \mathrm{tr}(P_1) + \mathrm{tr}(P_2) - 2\,\mathrm{tr}(P_1) + 2\,\mathrm{tr}(P_1 P_3) + 2\,\mathrm{tr}(P_1 P_4) < \varepsilon. \quad \blacksquare$$

corollary 2.3 *Let $\rho(t)$ be a continuous density-operator-valued function on an interval $I \subset \mathbb{R}$ and suppose that, for $t \in I$, $r_n(t)$ is an isolated eigenvalue of $\rho(t)$ with one-dimensional eigenspace. (This is equivalent to $r_{n+1}(t) < r_n(t) < r_{n-1}(t)$ if $2 \leq n \leq N-1$.) Then the corresponding eigenvector $\psi_n(t)$ can be chosen to be continuous.*

proof $\psi_n(t)$ is unique up to a phase factor. Choose $t_0 \in I$ and fix $\psi_n(t_0)$. By lemma 2.2, the phase-independent projection $|\psi_n(t)><\psi_n(t)|$ is continuous on I. $\psi_n(t)$ can be extended to an interval around t_0 by an expression of the form $\psi_n(t) = \dfrac{|\psi_n(t)><\psi_n(t)|\psi_n(t_0)>}{|<\psi_n(t)|\psi_n(t_0)>|}$, and similarly to the whole of I. $\blacksquare$

example 2.4 *Let $\rho(t)$, $t \in [0, 1]$, be a continuous path of density matrices on $\mathbb{C}^2$ with $\rho(0) = \begin{pmatrix} \frac{1}{2} + \varepsilon_0 & 0 \\ 0 & \frac{1}{2} - \varepsilon_0 \end{pmatrix}$ and $\rho(1) = \begin{pmatrix} \frac{1}{2} - \varepsilon_1 & 0 \\ 0 & \frac{1}{2} + \varepsilon_1 \end{pmatrix}$, where $0 < \varepsilon_0, \varepsilon_1 \leq \frac{1}{2}$.*

If $\rho(t)$ avoids the degeneracy point for all t then the eigenvector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ of $\rho(0)$ will move continuously to the eigenvector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ of $\rho(1)$.

Corollary 2.3 allows a complete description of the eigenvectors of a density operator in terms of continuous trajectories, provided that, at all times, all its eigenvalues are isolated and all its eigenspaces are one-dimensional. Such a density operator is never degenerate and both eigenvalues (by lemma 2.1) and eigenvectors (by corollary 2.3) will evolve continuously. This means that, according to the modal interpretation, the definite properties and the corresponding probabilities will also evolve continuously, although as mentioned in the introduction, this does not rule out the possibility, or even the necessity, of jumps for the properties actually possessed by an individual system. In the next section, we shall propose that avoiding degeneracy points is generic behaviour, at least for finite-dimensional systems.

3 The Co-Dimension of the Space of Degenerate Density Operators.

In this section, we shall demonstrate that, on a finite-dimensional Hilbert space, the space of degenerate density operators has co-dimension 3 in the space of all density operators. The state of a typical physical system will be subject to random environmental fluctuations, and can be expected to undergo some sort of local Brownian motion. This means that the state will change continuously and will have probability zero of ever hitting a degeneracy. It follows that, with probability one, the eigenvalues of the state are always isolated, and so, even without invoking the analyticity conditions to be introduced in the next section, the eigenvectors can be chosen to be continuous by corollary 2.3. As in example 2.4, these continuous eigenvectors always correspond to the same ordering of the eigenvalues. Thus, if $\rho(t) = \sum_{n=1}^{N} s_n(t)|\psi_n(t)\rangle\langle\psi_n(t)|$, where the $s_n(t)$ and $\psi_n(t)$ are continuous and if $s_1(0) > s_2(0) > \ldots > s_N(0)$, then, for all t, $s_1(t) > s_2(t) > \ldots > s_N(t)$.

definition 3.1 Let $\Sigma(\mathcal{H}_s)$ denote the set of all density operators on $\mathcal{H}_s$, a Hilbert space of dimension $N < \infty$. Let $\Sigma^n(\mathcal{H}_s)$ (respectively $\Sigma^d(\mathcal{H}_s)$) denote the subset of non-degenerate (resp. degenerate) density operators.

It follows from lemma 2.1 that $\Sigma^n(\mathcal{H}_s)$ is an open set in $\Sigma(\mathcal{H}_s)$. It is a dense set, because, if $\rho = \sum_{n=1}^{N} p_n|\psi_n\rangle\langle\psi_n|$ is an arbitary density operator, with $p_1 \geq p_2 \geq \ldots \geq p_N$, and $\sigma = \frac{1}{1-2^{-N}} \sum_{n=1}^{N} \frac{1}{2^n}|\psi_n\rangle\langle\psi_n|$, then $(1-x)\rho + x\sigma \in \Sigma^n(\mathcal{H}_s)$ for $0 < x \leq 1$ and $(1-x)\rho + x\sigma \to \rho$ as $x \to 0$.

The set of self-adjoint operators on $\mathcal{H}_s$ is a real vector space of dimension N^2, and $\Sigma(\mathcal{H}_s)$ is a subset of dimension $N^2 - 1$.

proposition 3.2 $\Sigma^d(\mathcal{H}_s)$ *has co-dimension 3 in* $\Sigma(\mathcal{H}_s)$.

proof Let $\Delta = (d_1, d_2, \ldots, d_{M(\Delta)})$ be a partition of N with $1 \leq d_1 \leq d_2 \leq \ldots \leq d_{M(\Delta)}$ and $d_1 + d_2 + \ldots + d_{M(\Delta)} = N$. Let Σ^Δ be the set of density matrices ρ with eigenvalues that can be partitioned according to Δ; so that, in other words, ρ has exactly $M(\Delta)$ distinct eigenvalues and these can be arranged so that exactly d_m eigenvalues have the m^{th} value. For example, $\Sigma^{(1,1,\ldots,1)} = \Sigma^n$.

Let G be the unitary group on $\mathcal{H}$. G is a compact Lie group of dimension N^2 with Lie algebra given by the self-adjoint matrices on $\mathcal{H}$. G acts on Σ by $U \cdot \rho = U\rho U^*$. Let G_ρ denote the stabilizer of ρ, so that $U \in G_\rho \iff U \cdot \rho = \rho$. G_ρ is a closed subgroup of G and, for $\rho \in \Sigma^\Delta$, G_ρ has dimension $\sum_{m=1}^{M(\Delta)} d_m^2$ so that G/G_ρ is a manifold of dimension $N^2 - \sum_{m=1}^{M(\Delta)} d_m^2$.

Let $\rho = \sum_{m=1}^{M(\Delta)} p_m P_m(\rho)$. Choose $\varepsilon > 0$ such that $m \neq m' \Rightarrow |p_m - p_{m'}| > 2\varepsilon$. Let $\mathcal{O} = \{\sigma : ||\rho - \sigma|| < \varepsilon\} \cap \Sigma^\Delta$.

Suppose that $\sigma \in \mathcal{O}$. By lemma 2.1, there is a unique ordering $(q_m)_{m=1}^{M(\Delta)}$ of the eigenvalues of σ so that $|p_m - q_m| < \varepsilon$, and then σ has a unique representation of the form $\sigma = \sum_{m=1}^{M(\Delta)} q_m P_m(\sigma)$.

There exists $U \in G$ such that $U^*\sigma U = \sum_{m=1}^{M(\Delta)} q_m P_m(\rho)$.

If σ has two such representations,

$$\sigma = U \sum_{m=1}^{M(\Delta)} q_m P_m(\rho) U^* = V \sum_{m=1}^{M(\Delta)} q_m P_m(\rho) V^*,$$

$$\text{then} \qquad V^*U \sum_{m=1}^{M(\Delta)} q_m P_m(\rho) U^*V = \sum_{m=1}^{M(\Delta)} q_m P_m(\rho),$$

so that $V^*U \in G_\rho$ and $U \in VG_\rho$.

Thus, $\mathcal{O}$ takes the form $\{U \sum_{m=1}^{M(\Delta)} s_m P_m(\rho) U^*\}$, where $s_m \geq 0$ and $|p_m - s_m| < \varepsilon$ for $k = 1, \ldots, M(\Delta)$, $\sum_{m=1}^{M(\Delta)} s_m = 1$, and U belongs to a neighbourhood of the identity in G/G_ρ. There are $M(\Delta) - 1$ dimensions of variation possible in the s_m, so that Σ^Δ is a manifold (with boundary) of dimension $N^2 - \sum_{m=1}^{M(\Delta)} d_m^2 + M(\Delta) - 1$.

$N^2 - \sum_{m=1}^{M(\Delta)} d_m^2 + M(\Delta) - 1 = N^2 - \sum_{m=1}^{M(\Delta)} (d_m^2 - 1) - 1$ so, as $d_m \geq 1$, $\dim(\Sigma^\Delta)$ is maximized at $N^2 - 1$ when $d_m = 1$ for all m, which is when $\Sigma^\Delta = \Sigma^{(1,1,\ldots,1)} = \Sigma^n$, and is next to maximal when $d_m = 1$ for $m = 1, \ldots, N-2$ and $d_{N-1} = 2$, for which $\dim(\Sigma^\Delta) = N^2 - 4$. ∎

4 Analyticity.

If a density operator $\rho(t)$ is an analytic function of t, then strong results on continuity of eigenfunctions are available.

definition 4.1 *Let $I \subset \mathbb{R}$ be an open interval. Suppose that $\rho(t)$ is a density operator on a Hilbert space $\mathcal{H}$ for $t \in I$. Let $\mathfrak{I}_1(\mathcal{H})$ denote the space of trace class operators on $\mathcal{H}$. We shall say that $\rho(t)$ is an analytic function on I if there is an open complex domain $D \supset I$ and an analytic function from D into $\mathfrak{I}_1(\mathcal{H})$ which agrees with $\rho(t)$ on I.*

We shall see below that if H is a Hamiltonian on a tensor product Hilbert space $\mathcal{H} = \mathcal{H}_s \otimes \mathcal{H}_e$ then there is a dense set of vectors $\Psi \in \mathcal{H}$ such that the reduced density operator $\rho(t) = (e^{-itH}|\Psi><\Psi|e^{itH})_s$ is analytic on $\mathbb{R}$.

theorem 4.2 *Let $\rho(t)$ be analytic on I and suppose that $t_0 \in I$. Let r be an isolated eigenvalue of $\rho(t_0)$ with K-dimensional eigenspace, where $K < \infty$. Then, there is an open interval $I_0 \subset I$ with $t_0 \in I_0$ on which there exist K (not necessarily distinct) numerical analytic functions $(r_k(t))_{k=1}^K$ and K vector-valued analytic functions $(\psi_k(t))_{k=1}^K$. $r_k(t_0) = r$ and, for each $t \in I_0$, $(\psi_k(t))_{k=1}^K$ is an orthonormal sequence of eigenvectors of $\rho(t)$ and $r_k(t)$ is the eigenvalue corresponding to $\psi_k(t)$.*

proof See[11], §1.1 and §2.2, or [12], §II.1, §II.4, §II.6, §VII.1 and §VII.3.

Kato's proof ([12]) involves finding, for some $M \leq K$, M distinct numerical analytic functions $(\hat{r}_m(t))_{m=1}^M$ on an interval I_0 containing t_0 which are such that $\hat{r}_m(t_0) = r$ and which, for each $t \in I_0$, are all the eigenvalues of $\rho(t)$ close to r, and then constructing a corresponding sequence $(\hat{P}_m^r(t))_{m=1}^M$ of projection-valued analytic functions on I_0 which, for each $t \in I_0$, form an orthogonal sequence of eigenprojections for $\rho(t)$ such that $\sum_{m=1}^M \dim \hat{P}_m^r(t) = K$. The $\psi_k(t)$ are constructed from the $\hat{P}_m^r(t)$. ∎

corollary 4.3

A) If $\dim \mathcal{H} = N < \infty$ then, on the whole interval I, there exist N numerical analytic functions $(r_n(t))_{n=1}^N$ and N vector-valued analytic functions $(\psi_n(t))_{n=1}^N$, such that, for each $t \in I$, $(\psi_n(t))_{n=1}^N$ is an orthonormal basis for $\mathcal{H}$ consisting of eigenvectors of $\rho(t)$ and $r_n(t)$ is the eigenvalue corresponding to $\psi_n(t)$.

B) If $\dim \mathcal{H} = \infty$ then each pair of functions $r_k(t)$ and $\psi_k(t)$ given by the theorem can be analytically continued at least until $r_k(t) \to 0$.

proof In case A, theorem 4.2 and its proof can be applied to any eigenvalue of $\rho(t)$. At all but a finite number of points of I_0, the functions $\hat{r}_m(t)$ constructed in the proof are distinct and the $\hat{P}_m^r(t)$ are the unique eigenprojections of $\rho(t)$ corresponding to those distinct eigenvalues. Thus, A is a consequence of the principle of uniqueness of analytic continuation for vector-valued functions ([15], §IX.4).

In case B, theorem 4.2 can be applied to any strictly positive eigenvalue. If $\dim \mathcal{H} = \infty$ then, of course, 0 cannot be an isolated eigenvalue with finite-dimensional eigenspace.

Suppose, in case B, that $r_k(t)$ is an analytic eigenvalue given by theorem 4.2 and that I_1 is the maximal interval containing t_0 to which it can be analytically extended. Suppose that $T = \sup I_1$ and that $T \in I$. Choose $\varepsilon > 0$. There is a finite subsequence $(r_{n,T})_{n=1}^N$ of the ordered eigenvalues of $\rho(T)$ such that $\sum_{n=1}^N r_{n,T} > 1 - \frac{1}{2}\varepsilon$. As N is finite, there is an open interval I_2 with $T \in I_2$ on which each of these eigenvalues has an analytic extension $r_{n,T}(t)$ which is an eigenvalue of $\rho(t)$ for $t \in I_2$ and such that $t \in I_2$ implies $\sum_{n=1}^N r_{n,T}(t) > 1 - \varepsilon$. By uniqueness of analytic continuation, none of these functions, which are extendable to T, can agree with $r_k(t)$ on $I_1 \cap I_2$. It follows that $r_k(t) < \varepsilon$ on $I_1 \cap I_2$ and so $r_k(t) \to 0$ as $t \to T$. ∎

Theorem 4.2 enables us to decompose $\rho(t)$ further, even at degeneracy points.

definition 4.4 *Let $\rho(t)$ be analytic on I and suppose that $t_0 \in I$. Call a projection P "assignable" for $\rho(t)$ at t_0 if and only if there exists a sequence $(t_n)_{n\geq 1} \subset I$ with $t_n \neq t_0$ for all n and $t_n \to t_0$ as $n \to \infty$ and there exists a sequence $(P_n(t_n))_{n\geq 1}$ of projections such that $P_n(t_n)$ is a spectral projection of $\rho(t_n)$ and $P_n(t_n)$ converges strongly to P as $n \to \infty$.*

Call a projection P "decomposition-assignable" for $\rho(t)$ at t_0 if either it is the projection onto the null space of $\rho(t_0)$ or if it is a minimal assignable projection such that $\mathrm{tr}(\rho(t_0)P) > 0$.

"Spectral projections" are the eigenprojections P_m of (1.1) or sums of such projections, and "minimal" is intended in the usual sense of the ordering of projections in Hilbert space.

corollary 4.5 *Let $\rho(t)$ be analytic on I and suppose that $t_0 \in I$.*

A) Let r be an isolated eigenvalue of $\rho(t_0)$ with finite-dimensional eigenspace $\mathcal{H}_r$. Let P^r be the projection onto $\mathcal{H}_r$.

Then P^r has a unique decomposition of the form $P^r = \sum_{m=1}^{M_r} \hat{P}_m^r$, where the $\hat{P}_m^r$ are decomposition-assignable projections for $\rho(t)$ at t_0.

B) Every assignable projection P for $\rho(t)$ at t_0 is a sum of projections which are either of the form $\hat{P}_m^r$ for some isolated eigenvalue of $\rho(t_0)$, or which satisfy $\rho(t_0)P = 0$.

C) Every spectral projection of $\rho(t_0)$ is assignable for $\rho(t)$ at t_0.

D) $\rho(t_0)$ has a unique decomposition of the form $\rho(t_0) = \sum_n q_n(t_0)Q_n(t_0)$, where the $Q_n(t_0)$ are an orthogonal family of decomposition-assignable projections satisfying $\sum_n Q_n(t_0) = 1$.

proof

A) The projections $\hat{P}_m^r$ are the $\hat{P}_m^r(t_0)$ constructed in the proof of theorem 4.2. They are assignable because $\hat{P}_m^r(t)$ is analytic and because, at all but a discrete set of points in the interval I_0, all of the functions in the sequence $(\hat{r}_m(t))_{m=1}^{M_r}$ differ. Their minimality and the uniqueness of the decomposition follow from the proof of B.

B) Let P be an assignable projection for $\rho(t)$ at t_0. Suppose that $(t_n)_{n\geq 1}$ and $(P_n(t_n))_{n\geq 1}$ are sequences with the properties required by definition 4.4.

Let r be an isolated eigenvalue of $\rho(t_0)$. Adopt the notation of the proof of theorem 4.2. For N sufficiently large, $(t_n)_{n\geq N} \subset I_0$. For $n \geq N$, either $P_n(t_n)\hat{P}^r_m(t_n) = \hat{P}^r_m(t_n)$ or $P_n(t_n)\hat{P}^r_m(t_n) = 0$. Convergence of $P_n(t_n)\hat{P}^r_m(t_n)$ requires that either $P\hat{P}^r_m(t_0) = \hat{P}^r_m(t_0)$ or $P\hat{P}^r_m(t_0) = 0$. To complete the proof, it is only necessary to show that, in infinite dimensions, P commutes with the projection onto the null space of $\rho(t_0)$. But this follows because the null space is the space orthogonal to the space spanned by the isolated eigenvectors.

C) Let P be a spectral projection of $\rho(t_0)$. P can be written either in the form $P = \sum_{r\in R} P^r$ or in the form $P = 1 - \sum_{r\in R} P^r$, where R is some set of isolated eigenvalues of $\rho(t_0)$ and the P^r are the corresponding projections. It is sufficient to consider the first case, because, if P is assignable then so is $1 - P$.

For each $r \in R$, P^r has a norm-continuous analytic extension $P^r(t)$ to some open interval I_r containing t_0, such that $P^r(t)$ is a spectral projection of $\rho(t)$. Let

$$P_n = \sum\{P^r(t_0 + \tfrac{1}{n}) : r \in R, t_0 + \tfrac{1}{n} \in I_r, \text{ and } ||P^r - P^r(t_0 + \tfrac{1}{n})|| < 2^{-N(r)}\},$$

where $N(r)$ is the number of elements of R larger than r. P_n is a spectral projection of $\rho(t_0 + \frac{1}{n})$. For any $r \in R$ there exists n_r such that $n \geq n_r$ implies $t_0 + \frac{1}{n} \in I_r$ and $||P^r - P^r(t_0 + \frac{1}{n})|| < 2^{-N(r)}$. Choose $\psi \in \mathcal{H}$ with $||\psi|| = 1$ and $\varepsilon > 0$. There exists N such that $||\sum\{P^r\psi : N(r) > N\}|| < \varepsilon$ and such that $2^{-N+1} < \varepsilon$. Let $m_1 = \sup\{n_r : N(r) \leq N\}$. Then $n \geq m_1$ implies

$$\begin{aligned}||(P - P_n)\psi|| &\leq ||\sum\{P^r\psi : N(r) > N\}|| + \sum\{||(P^r - P^r(t_0 + \tfrac{1}{n}))\psi|| : N(r) > N\} \\ &\quad + \sum\{||(P^r - P^r(t_0 + \tfrac{1}{n}))\psi|| : N(r) \leq N\} \\ &\leq 2\varepsilon + \sum\{||(P^r - P^r(t_0 + \tfrac{1}{n}))\psi|| : N(r) \leq N\}.\end{aligned}$$

But now $m_2 \geq m_1$ can be chosen sufficiently large that $n \geq m_2$ implies $\sum\{||(P^r - P^r(t_0 + \frac{1}{n}))\psi|| : N(r) \leq N\} < \varepsilon$. Then $n \geq m_2$ implies $||(P - P_n)\psi|| < 3\varepsilon$, and so P_n converges strongly to P.

D) This follows from A and B. ∎

We shall now establish conditions under which analyticity of $\rho(t)$ holds. Suppose that $\mathcal{H} = \mathcal{H}_s \otimes \mathcal{H}_e$. Let $(\psi_n)_{n\geq 1}$ be a basis for $\mathcal{H}_e$. Let $A \in \mathfrak{I}_1(\mathcal{H})$. Then the partial trace of A is defined as the unique operator A_s on $\mathcal{H}_s$ which satisfies

$$<\varphi|A_s|\varphi'> = \sum_{n\geq 1} <\varphi \otimes \psi_n|A|\varphi' \otimes \psi_n> \text{ for all } \varphi, \varphi' \in \mathcal{H}_s. \tag{4.6}$$

lemma 4.7 *For $A \in \mathfrak{I}_1(\mathcal{H})$, A_s exists and is independent of the basis $(\psi_n)_{n\geq 1}$. $A_s \in \mathfrak{I}_1(\mathcal{H}_s)$ and $||A_s||_1 \leq ||A||_1$.*

proof This is a well-known result (cf. [13] p. 382, problem 153a).

For $\varphi \in \mathcal{H}_s$ with $||\varphi|| = 1$, let P_φ be the projection of $\mathcal{H}$ onto the subspace $\varphi \otimes \mathcal{H}_e$. Then $P_\varphi A P_\varphi \in \mathfrak{I}_1(\mathcal{H})$ and

$$\operatorname{tr}(P_\varphi A P_\varphi) = \sum_{n \geq 1} <\varphi \otimes \psi_n | A | \varphi \otimes \psi_n>$$

is absolutely convergent and independent of the basis $(\psi_n)_{n\geq 1}$. It follows by the polarization identity that (4.6) defines an operator A_s on $\mathcal{H}_s$.

The second part of the lemma is a consequence of the facts that if $A \in \mathfrak{I}_1(\mathcal{H})$, then $||A||_1 = \sup\{|\operatorname{tr}(AB)| : ||B|| = 1\}$, and, if B is a bounded operator on $\mathcal{H}_s$, then $\operatorname{tr}(A_s B) = \operatorname{tr}(A(B \otimes I))$. ∎

corollary 4.8 *For $\Phi, \Psi \in \mathcal{H}$, $(|\Phi><\Psi|)_s$ exists and satisfies*

$$||(|\Phi><\Psi|)_s||_1 \leq ||\,|\Phi><\Psi|\,||_1 = ||\Phi||\,||\Psi||.$$

definition 4.9 *Let H be a self-adjoint operator on $\mathcal{H}$. For $T > 0$, a vector $\Psi \in \mathcal{H}$ is an analytic vector for H in $\{z : |z| < T\}$, ([17], p. 201) if Ψ is in the domain of H^n for all n and*

$$\sum_{n=0}^{\infty} \frac{||H^n \Psi||}{n!}(2T)^n < \infty.$$

If P_Ω are the spectral projections for H and $S < \infty$, then any $\Psi \in P_{[-S,S]}\mathcal{H}$ will be analytic for H in $\mathbb{C}$. Thus, for any Hamiltonian, there is a dense set of analytic vectors and also, every vector is analytic if H is bounded or if $\mathcal{H}$ is finite-dimensional.

theorem 4.10 *Let H be a self-adjoint operator on $\mathcal{H}$ and $\Psi \in \mathcal{H} = \mathcal{H}_s \otimes \mathcal{H}_e$ be an analytic vector for H in $D = \{z : |z| < T\}$. For $|z| < T$, write $\rho(z) = (e^{-izH}|\Psi><\Psi|e^{izH})_s$. Then $\rho(z)$ is an $\mathfrak{I}_1(\mathcal{H}_s)$-valued analytic function on D.*

proof $\rho(z)$ is analytic in D if and only if, for all $z \in D$,

$$\lim_{h \to 0} (\rho(z+h) - \rho(z))/h$$

exists in norm. Using corollary 4.8, this is a consequence of the analyticity of $e^{-izH}|\Psi>$. ∎

5 Examples.

The combination of theorems 4.2 and 4.10 gives a satisfactory description of the behaviour of the eigenvectors of the reduced state of a vector analytic for some Hamiltonian, except for the question of what may happen when an eigenvalue vanishes and the Hilbert space is infinite-dimensional. In this section, we address this question and we also show that discontinuous evolution of eigenvectors is possible if analyticity is not required. The first example shows that it is possible for eigenvectors to disappear when an eigenvalue vanishes, even if analyticity is assumed. Time reversal shows that eigenvectors can also appear. Example 5.1 is ultimately based on work of Rellich and Kato ([12], example V.4.14), but has been considerably adapted for the present context.

example 5.1 *For any $\theta > 0$, there is a Hilbert space $\mathcal{H} = \mathcal{H}_s \otimes \mathcal{H}_e$, a vector Ψ in $\mathcal{H}$ and a bounded Hamiltonian H on $\mathcal{H}$, such that the density operator $\rho(t) = (e^{-itH}|\Psi{>}{<}\Psi|e^{itH})_s$ has the following properties:*

$\rho(0)$ has a complete orthonormal set of eigenvectors $(\psi_n)_{n\geq 1}$. For each $m \geq 1$, there is a unique vector-valued analytic function $\psi_m(t)$ on $(-\infty, \frac{m\pi}{\theta})$ such that $\psi_m(0) = \psi_m$ and such that the sequence $(\psi_n(t))_{n\geq m}$ is a complete orthonormal set of eigenvectors for $\rho(t)$ when $\frac{(m-1)\pi}{\theta} \leq t < \frac{m\pi}{\theta}$

Thus, despite the fact that $\rho(t)$ is analytic — because H is bounded — any given eigenvector of $\rho(0)$ disappears in a finite time.

proof Let $\mathcal{H}_s = \mathcal{H}_e = L^2[0,1]$. Let $u_n(x) = \sqrt{2}\sin n\pi x$ and $u_0(x) = \sqrt{3}x$. u_0 is normalized and $(u_n)_{n\geq 1}$ is a complete orthonormal basis for $L^2[0,1]$.

Define $\Psi_A, \Psi_B \in \mathcal{H}_s \otimes \mathcal{H}_e$ by $\Psi_A = \sum_{n=1}^{\infty} \frac{\sqrt{90}}{n^2\pi^2}|u_n \otimes u_n{>}$, $\Psi_B = |u_0 \otimes u_0{>}$.

$||\Psi_A|| = ||\Psi_B|| = 1$.

Set $A = \sqrt{(|\Psi_A{>}{<}\Psi_A|)_s} = \sum_{n=1}^{\infty} \frac{\sqrt{90}}{n^2\pi^2}|u_n{>}{<}u_n|$ and $B = (|\Psi_B{>}{<}\Psi_B|)_s = |u_0{>}{<}u_0|$.

The vectors Ψ_A and $\dfrac{\Psi_B - {<}\Psi_A|\Psi_B{>}\Psi_A}{\sqrt{1 - |{<}\Psi_A|\Psi_B{>}|^2}} = \sqrt{\frac{5}{3}}\Psi_B - \sqrt{\frac{2}{3}}\Psi_A$ are orthogonal and normalized, so, for $\theta \in \mathbb{R}$, there exists a bounded Hamiltonian H on $\mathcal{H}_s \otimes \mathcal{H}_e$ such that

$$e^{-itH}\Psi_A = (\cos\theta t - \sqrt{\frac{2}{3}}\sin\theta t)\Psi_A + \sqrt{\frac{5}{3}}\sin\theta t\,\Psi_B.$$

$$(e^{-itH}|\Psi_A{>}{<}\Psi_A|e^{itH})_s = ((\cos\theta t - \sqrt{\frac{2}{3}}\sin\theta t)A + \sqrt{\frac{5}{3}}\sin\theta t\,B)^2.$$

It is possible to give a complete analysis of the eigenvectors and eigenvalues of any operator on $L^2[0,1]$ of the form $(aA + bB)^2$ for $a, b \in \mathbb{R}$. To do this, it is convenient to set $A' = A/\sqrt{90}$. Let $\varphi \in L^2[0,1]$ be an (unnormalized) eigenvector of $A' + bB$ satisfying $(A' + bB)\varphi = \lambda\varphi$. Suppose that $\varphi = \sum_{n=1}^{\infty} c_n u_n$.

If $\lambda = 0$ then $A'\varphi = -bB\varphi$ and so

$$\frac{1}{n^2\pi^2} c_n = -b < u_n|u_0 > < u_0|\varphi > = -\sqrt{6}b\frac{(-1)^{n+1}}{n\pi} < u_0|\varphi >.$$

This is impossible because either $< u_0|\varphi > \neq 0$, in which case $c_n \sim n$, or $< u_0|\varphi > = 0$, in which case $c_n = 0$ for all n. Thus $\lambda \neq 0$.

$$\begin{aligned}\lambda^2\varphi &= \lambda(A' + bB)\varphi = (A' + bB)^2\varphi \\ &= \sum_{n=1}^{\infty} \frac{c_n}{n^4\pi^4} u_n + b(A'u_0 < u_0|\varphi > + u_0 < u_0|A'|\varphi >) + b^2 u_0 < u_0|\varphi >.\end{aligned}$$

Thus, $$\lambda^2\varphi(x) = \sqrt{2}\sum_{n=1}^{\infty} \frac{c_n}{n^4\pi^4} \sin n\pi x$$
$$+b(-\frac{1}{2\sqrt{3}}(x^3 - x) < u_0|\varphi > + \sqrt{3}x < u_0|A'|\varphi >) + \sqrt{3}b^2 x < u_0|\varphi >.$$

The first term and its first and second derivatives are uniformly convergent in x. Thus φ is twice continuously differentiable, and

$$\lambda^2\varphi''(x) = -\sqrt{2}\sum_{n=1}^{\infty} \frac{c_n}{n^2\pi^2} \sin n\pi x - b\sqrt{3}x < u_0|\varphi > = -((A' + bB)\varphi)(x) = -\lambda\varphi(x).$$

Also $\varphi(0) = 0$.

$\lambda\varphi''(x) = -\varphi(x)$, $\varphi(0) = 0$ has solutions $\varphi(x) = \sin \mu x$ with $\mu = 1/\sqrt{\lambda}$ for $\lambda > 0$, and $\varphi(x) = \sinh \mu x$ with $\mu = 1/\sqrt{-\lambda}$ for $\lambda < 0$. As $\sin \mu x = -\sin(-\mu x)$ and $\sinh \mu x = -\sinh(-\mu x)$ we may take $\mu > 0$.

φ satisfies $< u_0|(A' + bB - \lambda)|\varphi > = 0$. For $\varphi(x) = \sinh \mu x$ this is equivalent to

$$\sinh \mu = 3b(\sinh \mu - \mu \cosh \mu), \tag{5.2}$$

and for $\varphi(x) = \sin \mu x$ it is equivalent to

$$\sin \mu = 3b(\sin \mu - \mu \cos \mu). \tag{5.3}$$

It is straightforward to check that any solution either to equation (5.2) or to equation (5.3) does correspond to an eigenvector of $A' + bB$. This means that, for any b, we can identify the set of all eigenvectors of $A' + bB$ and, as $A' + bB$ is a compact operator, this set will be complete ([14], theorem VI.16).

There are then two distinct cases. If $b \geq 0$ then there is no solution to (5.2). There is a solution to (5.3) with $0 < \mu \leq \pi$ which we shall label as $\mu_1(b)$ as well as a solution which we shall label as $\mu_{n+1}(b)$ in each interval $n\pi < \mu \leq (n+1)\pi$. For $b < 0$, there is a unique solution $\mu_0(b)$ to (5.2). As $b \nearrow 0^-$, $\mu_0(b) \to \infty$. There is also a single solution $\mu_n(b)$ to (5.3) in each interval $n\pi < \mu < (n+1)\pi$ with $n = 1, 2, \ldots$, but there is no solution with $0 < \mu < \pi$.

Write $\varphi_n(b) = \varphi_n(b, x) = \sqrt{\dfrac{4\mu_n(b)}{2\mu_n(b) - \sin 2\mu_n(b)}} \sin \mu_n(b)x$ for $n = 1, 2, \ldots$ and write

$$\varphi_0(b) = \varphi_0(b, x) = \sqrt{\frac{4\mu_0(b)}{\sinh 2\mu_0(b) - 2\mu_0(b)}} \sinh \mu_0(b)x.$$

If $b \geq 0$ then $A' + bB$ is positive, all its eigenvalues are positive, and a complete orthonormal eigenbasis is given by $(\varphi_n(b))_{n\geq 1}$.

For $b < 0$ a complete orthonormal eigenbasis is given by $(\varphi_n(b))_{n\geq 1} \cup \{\varphi_0(b)\}$.

It follows from the general, textbook, version of theorem 4.2, that for $n \geq 1$ the functions $\mu_n(b)$ are analytic in b on $\mathbb{R}$ as are the corresponding eigenvectors $\varphi_n(b)$, and also that $\mu_0(b)$ and $\varphi_0(b)$ are analytic in b on $(-\infty, 0)$.

It is also necessary to follow the eigenfunctions of $aA' + B = a(A' + \frac{1}{a}B)$ through $a = 0$. In fact, as a increases through 0, $\varphi_n(\frac{1}{a})$ continues analytically to $\varphi_{n+1}(\frac{1}{a})$ for $n \geq 0$.

Now careful tracing of the eigenfunctions of $\rho(t)$ yields the required result. ∎

In this example, $\rho(t)$ is periodic with period $2\pi/\theta$, but none of the eigenvectors is periodic. This demonstrates that eigenvectors may fail to reflect fundamental physical properties of density operators.

Many variations of example 5.1 are possible:

example 5.4 *For any $\theta > 0$, there is a vector Ψ in a Hilbert space $\mathcal{H} = \mathcal{H}_s \otimes \mathcal{H}_e$ and a bounded Hamiltonian H on $\mathcal{H}$ such that the density operator $\rho(t) = (e^{-itH}|\Psi\rangle\langle\Psi|e^{itH})_s$ has the following properties:*

$\rho(t)$ is periodic with period $2\pi/\theta$. For each $n \geq 1$, there is a vector-valued analytic function $\psi_n(t)$ on $\mathbb{R}$ which is also periodic with period $2\pi/\theta$, and there is a vector-valued analytic function $\psi_0(t)$ on $(\frac{\pi}{\theta}, \frac{2\pi}{\theta})$ such that, for every integer m, a complete orthonormal set of eigenvectors for $\rho(t)$ is given by $(\psi_n(t))_{n\geq 1}$ when $\frac{2m\pi}{\theta} \leq t \leq \frac{(2m+1)\pi}{\theta}$ and by $(\psi_n(t))_{n\geq 1} \cup \{\psi_0(t - \frac{(2m-2)\pi}{\theta})$ when $\frac{(2m-1)\pi}{\theta} < t < \frac{2m\pi}{\theta}$.

Thus, in this case, $\rho(t)$ has eigenvectors which appear and then disappear.

proof Let $\mathcal{H}_s = \mathcal{H}_e = \mathcal{H}_a \oplus L^2[0, 1]$, where $\mathcal{H}_a$ is a one-dimensional space with basis vector χ_0. Let $C = |\chi_0\rangle\langle\chi_0|$. Let $\mathcal{H} = \mathcal{H}_s \otimes \mathcal{H}_e$.

Define Ψ_A and Ψ_B as in example 5.1.

Define $\Psi_C = |\chi_0 \otimes \chi_0\rangle$.

$$(|\Psi_C\rangle\langle\Psi_C|)_s = C \text{ and } (|\Psi_C\rangle\langle\Psi_A|)_s = |\Psi_C\rangle\langle\Psi_B|)_s = 0.$$

For $a, b, c \in \mathbb{R}$, $(|a\Psi_A + b\Psi_B + c\Psi_C\rangle\langle a\Psi_A + b\Psi_B + c\Psi_C|)_s = (aA + bB)^2 + c^2C.$

For $\theta \in \mathbb{R}$, there exists a bounded Hamiltonian H on $\mathcal{H}$, with $||H|| = \theta$, such that $e^{-itH}\Psi_A = \Psi_A$, and

$$e^{-itH}\Psi_C = \cos\theta t\,\Psi_C + \sin\theta t(\sqrt{\frac{5}{3}}\Psi_B - \sqrt{\frac{2}{3}}\Psi_A).$$

Let $\Psi = \frac{1}{\sqrt{2}}\Psi_A + \frac{1}{\sqrt{2}}\Psi_C$.

$$e^{-itH}\Psi = (\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}}\sin\theta t)\Psi_A + \sqrt{\frac{5}{6}}\sin\theta t\,\Psi_B + \frac{1}{\sqrt{2}}\cos\theta t\,\Psi_C.$$

$$\rho(t) = (e^{-itH}|\Psi\rangle\langle\Psi|e^{itH})_s = ((\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}}\sin\theta t)A + \sqrt{\frac{5}{6}}\sin\theta t\,B)^2 + \frac{1}{2}\cos^2\theta t\,C.$$

The eigenvector analysis now follows example 5.1, using the fact that $\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}}\sin\theta t > 0$ for all t. ∎

example 5.5 *Let $(t_n)_{n\geq 1}$ be any sequence of real numbers (for example, some counting of the rational numbers). Then there is a vector Φ in a Hilbert space $\mathcal{K} = \mathcal{K}_s \otimes \mathcal{K}_e$ and a bounded Hamiltonian K on $\mathcal{K}$ such that the density operator $\sigma(t) = (e^{-itK}|\Phi\rangle\langle\Phi|e^{itK})_s$ has an eigenvector disappearing at each point of the sequence $(t_n)_{n\geq 1}$.*

proof Let $\mathcal{H}_s$, $\mathcal{H}_e$, H, and Ψ be as in example 5.1. For each $n \geq 1$, let $\mathcal{H}_s^n$ be an isomorphic copy of $\mathcal{H}_s$ and let $\mathcal{H}_e^n$ be an isomorphic copy of $\mathcal{H}_e$. Let $\mathcal{K}_s = \oplus_{n=1}^{\infty}\mathcal{H}_s^n$ and $\mathcal{K}_e = \oplus_{n=1}^{\infty}\mathcal{H}_e^n$. Define a bounded Hamiltonian K on $\mathcal{K} = \mathcal{K}_s \otimes \mathcal{K}_e$ by $K = \oplus_{n=1}^{\infty}H_n$, where H_n is the copy of H on $\mathcal{H}_s^n \otimes \mathcal{H}_e^n$. Define $\Phi = \sum_{n=1}^{\infty}\frac{1}{2^{n/2}}e^{it_nH_n}\Psi_n$, where Ψ_n is the copy of Ψ on $\mathcal{H}_s^n \otimes \mathcal{H}_e^n$. Then $\sigma(t) = (e^{-itK}|\Phi\rangle\langle\Phi|e^{itK})_s = \oplus_{n=1}^{\infty}\frac{1}{2^n}\rho_n(t - t_n)$, where ρ_n is the copy of ρ on $\mathcal{H}_s^n$. $\rho_n(t - t_n)$ has an eigenvector disappearing at t_n. ∎

When we work with unbounded Hamiltonians in quantum mechanics, it is usually necessary to place some restriction on the wave-function; at the very least, that it have bounded expected energy. Analyticity (definition 4.9) is a comparatively strong restriction, but the final example of this section shows that it may be required if the modal interpretation is to avoid the possibility of discontinuous definite properties. In fact, with an appropriate choice of total space $\mathcal{H}$ and Hamiltonian H, we can obtain the situation envisaged in examples 1.3 and 1.4.

example 5.6 *There is a Hamiltonian H on a Hilbert space $\mathcal{H} = \mathcal{H}_s \otimes \mathcal{H}_e$ and a vector Ψ in $\mathcal{H}$ which is in the domain of H^n for all n, such that for no $\varepsilon > 0$ is there any* *continuous vector-valued function* $\psi(t)$ *which is a normalized eigenvector of the density operator* $\rho(t) = (e^{-itH}|\Psi><\Psi|e^{itH})_s$ *for all* $t \in [-\varepsilon, \varepsilon]$.

proof Let $\mathcal{H} = L^2(\mathbb{R})^4 = \{\varphi = (\varphi_1, \varphi_2, \varphi_3, \varphi_4) : \varphi_i \in L^2(\mathbb{R})\}$ with

$$||\varphi||^2 = \sum_{i=1}^{4} \int_{\mathbb{R}} |\varphi_i(x)|^2 dx.$$

Define $H = (-i\frac{d}{dx}, i\frac{d}{dx}, -i\frac{d}{dx}, i\frac{d}{dx})$ on the standard domain so that $(e^{-itH}\varphi)(x) = (\varphi_1(x-t), \varphi_2(x+t), \varphi_3(x-t), \varphi_4(x+t))$.

$\mathcal{H}$ is isomorphic to $\mathbb{C}^2 \otimes (L^2(\mathbb{R}) \oplus L^2(\mathbb{R}))$ under an isomorphism which sends $(\varphi_1, \varphi_2, \varphi_3, \varphi_4)$ to $\begin{pmatrix} (\varphi_1, \psi_1) \\ (\varphi_2, \psi_2) \end{pmatrix}$, where $\psi_1(x) = \chi_{(-\infty,0]}(x)\varphi_4(x) + \chi_{[0,\infty)}(x)\varphi_3(x)$ and $\psi_2(x) = \chi_{(-\infty,0]}(x)\varphi_3(x) + \chi_{[0,\infty)}(x)\varphi_4(x)$.

If $\Psi = \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} (\varphi_1, \psi_1) \\ (\varphi_2, \psi_2) \end{pmatrix}$, then

$$(|\Psi><\Psi|)_s = \begin{pmatrix} \int_{\mathbb{R}} |\varphi_1(x)|^2 dx + \int_{\mathbb{R}} |\psi_1(x)|^2 dx & \int_{\mathbb{R}} \varphi_1(x)\bar{\varphi}_2(x) dx + \int_{\mathbb{R}} \psi_1(x)\bar{\psi}_2(x) dx \\ \int_{\mathbb{R}} \bar{\varphi}_1(x)\varphi_2(x) dx + \int_{\mathbb{R}} \bar{\psi}_1(x)\psi_2(x) dx & \int_{\mathbb{R}} |\varphi_2(x)|^2 dx + \int_{\mathbb{R}} |\psi_2(x)|^2 dx \end{pmatrix}.$$

To see this, consider, for example, the 1-2 component of $(|\Psi><\Psi|)_s$. This is given by $\sum_{n=1}^{\infty} <\xi_n|f><g|\xi_n>$, where $(\xi_n)_{n\geq 1}$ is a basis for $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$. Let $(\eta_n)_{n\geq 1}$ be a basis for $L^2(\mathbb{R})$ and set $\xi_{2n} = (\eta_n, 0)$, $\xi_{2n+1} = (0, \eta_n)$.

Then $\displaystyle\sum_{n=1}^{\infty} <\xi_n|f><g|\xi_n>$

$$= \sum_{n=1}^{\infty} \int_{\mathbb{R}} \bar{\eta}_n(x)\varphi_1(x) dx \int_{\mathbb{R}} \bar{\varphi}_2(x)\eta_n(x) dx + \sum_{n=1}^{\infty} \int_{\mathbb{R}} \bar{\eta}_n(x)\psi_1(x) dx \int_{\mathbb{R}} \bar{\psi}_2(x)\eta_n(x) dx.$$

Now let u be a C^∞ function such that $u(x) = 0$ unless $x \in (0,1)$, $u(x) > 0$ for $x \in (0,1)$, and $\int_0^1 u(x)^2 dx = 1$. Set

$$\Psi(x) = (\varphi_1(x), \varphi_2(x), \varphi_3(x), \varphi_4(x)) = (\tfrac{1}{\sqrt{2}}u(x), \tfrac{1}{2}u(x+1), \tfrac{1}{2}u(x+1), 0).$$

$||\Psi||^2 = 1$.

$$(e^{-itH}|\Psi\rangle\langle\Psi|e^{itH})_s = \begin{pmatrix} \frac{1}{2}+\frac{1}{4}\int_0^\infty |u(x+1-t)|^2dx & \frac{1}{2\sqrt{2}}\int_{\mathbb{R}} u(x-t)u(x+1+t)dx \\ \frac{1}{2\sqrt{2}}\int_{\mathbb{R}} u(x-t)u(x+1+t)dx & \frac{1}{4}+\frac{1}{4}\int_{-\infty}^0 |u(x+1-t)|^2dx \end{pmatrix}.$$

$\int_0^\infty |u(x+1-t)|^2dx > 0$ and $\int_{\mathbb{R}} u(x-t)u(x+1+t)dx = 0$ for $t > 0$, while $\int_0^\infty |u(x+1-t)|^2dx = 0$ and $\int_{\mathbb{R}} u(x-t)u(x+1+t)dx > 0$ for $-1 < t < 0$.

$(e^{-itH}|\Psi\rangle\langle\Psi|e^{itH})_s$ can now be analysed using example 1.3. ∎

6 An Extension of the Modal Interpretation.

The conventional modal interpretation uses the spectral resolution (1.1) of the reduced state ρ to attribute definite properties to the system S; proposing that at time t system S possesses P_m with probability $p_m \dim(P_m)$. This property attribution suffers from discontinuities at degeneracy points, where in particular the dimension of the definite properties changes.

Theorem 4.10 suggests that it may not be unreasonable to assume that the state $\rho(t)$ of a subsystem is an analytic function of time. In that case, we can use the analyticity properties to extend the conventional rule for property assignment. Recall from corollary 4.5D, that if $\rho(t)$ is analytic at t_0 then $\rho(t_0)$ has a unique decomposition of the form

$$\rho(t_0) = \sum_n q_n(t_0)Q_n(t_0), \tag{6.1}$$

where the $Q_n(t_0)$ are an orthogonal family of decomposition-assignable projections satisfying $\sum_n Q_n(t_0) = 1$. (6.1) can be used to attribute properties to the system S, if we claim that, at time t_0, system S possesses $Q_n(t_0)$ with probability $q_n(t_0)\dim(Q_n(t_0))$. This rule of property attribution defines an extended modal interpretation, because the decomposition (6.1) is finer than the spectral resolution (1.1).

This extension contradicts a number of discussions in which the main rule of the conventional modal interpretation is uniquely derived from certain postulates ([6], [18]). We escape these uniqueness results, because our extended modal interpretation violates the postulate that the set of definite properties should be determined solely by the reduced state of the system at a fixed time. In our extension, the definite properties are determined by the reduced state seen as a dynamically evolving object. We think this postulate is just as reasonable.

The advantages of such an extension are twofold: first, degeneracies no longer represent exceptional points, where something dramatic happens with the definite properties; second, continuous trajectories can be used in formulating general proposals for dynamics in the modal interpretation.

In the most straightforward cases, the $Q_n(t)$ of (6.1) can be chosen to be continuous in t on the entire interval on which ρ is defined. Corollary 4.3A and theorem 4.10 show that this is always possible for a subsystem of a finite-dimensional Hilbert space $\mathcal{H}$ on which the dynamics is Hamiltonian. Corollary 2.3 shows that an analogous result holds whenever degeneracies are avoided, at least if we require in addition that the null space of ρ be empty if the system $\mathcal{H}_s$ is infinite-dimensional.

In more general cases, we must be more cautious. Example 5.6 demonstrates that an assumption of analyticity may be required, even given Hamiltonian dynamics and a finite-dimensional subsystem. Indeed, we have restricted the definition of assignability to cases in which $\rho(t)$ is analytic precisely because of such cases. If $\mathcal{H}_s$ is infinite-dimensional and $\rho(t)$ is analytic on an interval I, then by corollary 4.5D there is at each $t_0 \in I$ a unique decomposition of the form (6.1) in which the $Q_n(t_0)$ are orthogonal decomposition-assignable projections. However, theorem 4.2 only shows that for each Q_n in the decomposition (with $q_n(t_0) \neq 0$) there is an open interval I_n containing t_0, such that an analytic function $Q_n(t)$ can be defined on I_n. $Q_n(t)$ cannot in general be extended further than a point at which $q_n(t)$ tends to zero. As shown in examples 5.1 and 5.4, $Q_n(t)$ can be born or die at such points. As example 5.5 shows, there may be no open interval I containing t_0 such that all $Q_n(t)$ are analytically extendable to the whole of I. Thus, at any one time, we can write down a decomposition (6.1), but since trajectories can be born or die, there is in general no labelling of the projections such that for two different times t_1 and t_2 and for all n, $Q_n(t_1)$ and $Q_n(t_2)$ can be connected by an analytic trajectory.

Whenever $\rho(t)$ is analytic, the functions $q_n(t)$, with $q_n(t) > 0$, are distinct analytic functions on their intervals of definition. This means that they can agree only at isolated instants: they can only cross, and not split or merge. Such degeneracies, which might be called *passing*, do not lead to discontinuities in the corresponding $Q_n(t)$. What happens to $Q_n(t)$ as $q_n(t) \to 0$ or to the null projection of $\rho(t)$ is of no physical relevance, as such projections are possessed with arbitrarily small, or zero, probability. If a function $Q_n(t)$ is multi-dimensional, one might say that the corresponding eigenvalue $q_n(t)$ is *permanently* degenerate. While the extended modal interpretation and the conventional modal interpretation agree in their treatment of permanent degeneracies, they differ in their treatment of passing degeneracies.

In the cases in which continuity of trajectories holds, one can use the continuous trajectories to provide a framework for studying the dynamics of the actually possessed properties in the modal interpretation. As noted in the introduction, continuity of assignable projections does not entail continuity of actually possessed properties. Nevertheless, the two can be related by formulating the general evolution of the actually possessed properties in terms of stochastic transitions from one continuous trajectory to another. Transition probabilities in this sense have been derived by Vermaas [19] in special cases. General proposals for dynamics along these lines, and which comply with the results by Vermaas, will be the subject of [20].

7 Instability.

The examples in this section show that, regardless of how well we can control the initial state of some physical system, radical alterations in the definite properties defined by the restriction of that state to a subsystem can be caused by arbitrarily small changes either in an external parameter or in the identification of the subsystem. As these conclusions relate to behaviour *near* a degeneracy, they are physically relevant despite the proposal in section 3 that degeneracy points are almost never actually hit, and despite the continuity results proved in section 4.

For instability problems, it is not necessary to consider infinite-dimensional spaces. We shall work on $\mathbb{C}^4$. An isomorphism of $\mathbb{C}^4$ to $\mathbb{C}^2 \otimes \mathbb{C}^2$ can be specified by identifying an orthonormal basis $(f_i)_{i=1}^4$ of $\mathbb{C}^4$ with the sequence $(e_1 \otimes e_1, e_2 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_2)$, where (e_1, e_2) is a basis for $\mathbb{C}^2$.

Under this isomorphism, if a pure state Ψ has components $\Psi = \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix}$ in the given basis of $\mathbb{C}^4$, then $(|\Psi\rangle\langle\Psi|)_s$ has components $\begin{pmatrix} |\alpha|^2 + |\gamma|^2 & \alpha\bar{\beta} + \gamma\bar{\delta} \\ \beta\bar{\alpha} + \delta\bar{\gamma} & |\beta|^2 + |\delta|^2 \end{pmatrix}$ in the given basis of $\mathbb{C}^2$.

Choose $\varepsilon > 0$. The density matrices ρ_ε and σ_ε of example 1.3 are given by taking Ψ_{ρ_ε} with co-ordinates

$$\alpha = \sqrt{\tfrac{1}{2} + \varepsilon}, \quad \beta = \gamma = 0, \quad \text{and} \quad \delta = \sqrt{\tfrac{1}{2} - \varepsilon},$$

and by taking $\Psi_{\sigma_\varepsilon}$ with co-ordinates

$$\alpha = \delta = \tfrac{1}{2}\left(\sqrt{\tfrac{1}{2} + \varepsilon} + \sqrt{\tfrac{1}{2} - \varepsilon}\ \right) \quad \text{and} \quad \beta = \gamma = \tfrac{1}{2}\left(\sqrt{\tfrac{1}{2} + \varepsilon} - \sqrt{\tfrac{1}{2} - \varepsilon}\ \right).$$

For ε sufficiently small, these vectors are arbitrarily close, so that arbitrarily small environmental perturbations can move one to the other.

example 7.1 *There exists a Hamiltonian $H(\eta)$ on a Hilbert space $\mathcal{H} = \mathcal{H}_s \otimes \mathcal{H}_e$ and a vector $\Psi \in \mathcal{H}$ such that $H(\eta)$ is bounded and depends analytically on the parameter η and $(e^{-itH(\eta)}|\Psi\rangle\langle\Psi|e^{itH(\eta)})_s$ is jointly analytic in t and η. However, there exist t_0 and η_0 such that, for any $\varepsilon > 0$, there exist t_1, t_2, η_1, and η_2, with*

$$|t_1 - t_0| + |t_2 - t_0| + |\eta_1 - \eta_0| + |\eta_2 - \eta_0| < \varepsilon \tag{7.2}$$

and $||\xi - \xi'|| > \frac{1}{2}$ *for any pair* (ξ, ξ') *consisting of an eigenvector* ξ *of* $(e^{-it_1 H(\eta_1)}|\Psi\rangle\langle\Psi|e^{it_1 H(\eta_1)})_s$ *and an eigenvector* ξ' *of* $(e^{-it_2 H(\eta_2)}|\Psi\rangle\langle\Psi|e^{it_2 H(\eta_2)})_s$.

proof Choose $\eta \in [0, 2\pi)$. An orthonormal basis for $\mathbb{C}^4$ is given by

$$\psi^1_\eta = \begin{pmatrix} \cos\eta \\ 0 \\ 0 \\ \sin\eta \end{pmatrix} \quad \psi^2_\eta = \begin{pmatrix} \sin\eta \\ 0 \\ 0 \\ -\cos\eta \end{pmatrix} \quad \psi^3 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \psi^4 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

and a Hamiltonian $H(\eta)$ which depends analytically on η can be defined by $H(\eta) = i(|\psi^2_\eta\rangle\langle\psi^3| - |\psi^3\rangle\langle\psi^2_\eta|)$, so that

$$\begin{aligned} e^{-itH(\eta)} = |\psi^1_\eta\rangle\langle\psi^1_\eta| + \cos t(|\psi^2_\eta\rangle\langle\psi^2_\eta| + |\psi^3\rangle\langle\psi^3|) \\ + \sin t(|\psi^2_\eta\rangle\langle\psi^3| - |\psi^3\rangle\langle\psi^2_\eta|) + |\psi^4\rangle\langle\psi^4|. \end{aligned}$$

Let $\Psi = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$.

$$\begin{aligned} e^{-itH(\eta)}\Psi &= \cos\eta|\psi^1_\eta\rangle + \sin\eta\cos t|\psi^2_\eta\rangle - \sin\eta\sin t|\psi^3_\eta\rangle \\ &= \begin{pmatrix} \cos^2\eta + \sin^2\eta\cos t \\ -\sin\eta\sin t \\ 0 \\ \sin\eta\cos\eta(1-\cos t) \end{pmatrix} = \begin{pmatrix} 1 - \sin^2\eta(1-\cos t) \\ -\sin\eta\sin t \\ 0 \\ \sin\eta\cos\eta(1-\cos t) \end{pmatrix}. \end{aligned}$$

$$\begin{aligned} \text{Define} \quad \rho(t,\eta) &= (e^{-itH(\eta)}|\Psi\rangle\langle\Psi|e^{itH(\eta)})_s \\ &= \begin{pmatrix} (1-\sin^2\eta(1-\cos t))^2 & -\sin\eta\sin t(1-\sin^2\eta(1-\cos t)) \\ -\sin\eta\sin t(1-\sin^2\eta(1-\cos t)) & \sin^2\eta\cos^2\eta(1-\cos t)^2 + \sin^2\eta\sin^2 t \end{pmatrix} \\ &= \begin{pmatrix} (1-2\sin^2\eta\sin^2\frac{1}{2}t)^2 & -\sin\eta\sin t(1-2\sin^2\eta\sin^2\frac{1}{2}t) \\ -\sin\eta\sin t(1-2\sin^2\eta\sin^2\frac{1}{2}t) & 4\sin^2\eta\cos^2\eta\sin^4\frac{1}{2}t + \sin^2\eta\sin^2 t \end{pmatrix}. \end{aligned}$$

$\rho(\pi,\eta) = \begin{pmatrix} \cos^2 2\eta & 0 \\ 0 & \sin^2 2\eta \end{pmatrix}$, so that $\rho(\pi, \frac{\pi}{8}) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$.

$\sin^2\frac{\pi}{8} = \frac{1}{2}(1-\cos\frac{\pi}{4}) = \frac{1}{2}(1-\frac{1}{\sqrt{2}})$. $\cos^2\frac{\pi}{8} = \frac{1}{2}(1+\frac{1}{\sqrt{2}})$.

For t close to π, define $\eta(t) = \sin^{-1}\left(\frac{\sin\frac{\pi}{8}}{\sin\frac{1}{2}t}\right)$. $\sin\eta(t)\sin\frac{1}{2}t = \sin\frac{\pi}{8}$, so that

$$\rho(t,\eta(t)) = \begin{pmatrix} \frac{1}{2} & -\sqrt{(1-\frac{1}{\sqrt{2}})}\cos\frac{1}{2}t \\ -\sqrt{(1-\frac{1}{\sqrt{2}})}\cos\frac{1}{2}t & \frac{1}{2} \end{pmatrix}.$$

The conditions required can be satisfied by taking $\eta_0 = \frac{\pi}{8}$, $t_0 = t_1 = \pi$, and $\eta_2 = \eta(t_2)$. $t_2 > \pi$ and $\eta_1 > \frac{\pi}{8}$ are to be chosen so that (7.2) is satisfied. ∎

By theorems 4.2 and 4.10, for each η, the eigenvectors of $(e^{-itH(\eta)}|\Psi\rangle\langle\Psi|e^{itH(\eta)})_s$ in example 7.1 can be chosen to be analytic functions of t. However the rate at which these functions change with t becomes arbitrarily large as η approaches η_0.

We can also use the vectors Ψ_{ρ_ε} and $\Psi_{\sigma_\varepsilon}$ to show that the definite properties of a subsystem can depend with arbitrary sensitivity on the precise identification of that subsystem. This is a serious problem, because at least at the level of relativistic quantum field theory [21], there does not seem any reason to believe that there is any splitting of the world into subsystems of the kind considered that reflects a natural underlying observer-independent and state-independent splitting.

In general, an isomorphism V from a Hilbert space $\mathcal{H}$ of dimension N_sN_e to a tensor product $\mathcal{H}_s \otimes \mathcal{H}_e$, where $\dim \mathcal{H}_s = N_s$ and $\dim \mathcal{H}_e = N_e$ can be determined simply by choosing a suitably indexed basis $(\chi_{mn})_{m=1}^{N_s}{}_{n=1}^{N_e}$ for $\mathcal{H}$ and defining V by $V\chi_{mn} = \varphi_m \otimes \psi_n$, where $(\varphi_m)_{m=1}^{N_s}$ is a given basis for $\mathcal{H}_s$ and $(\psi_n)_{n=1}^{N_e}$ is a given basis for $\mathcal{H}_e$.

Holding $(\varphi_m)_{m=1}^{N_s}$ and $(\psi_n)_{n=1}^{N_e}$ fixed, let $(\chi_{mn})_{m=1}^{N_s}{}_{n=1}^{N_e}$ and $(\chi'_{mn})_{m=1}^{N_s}{}_{n=1}^{N_e}$ be different bases for $\mathcal{H}$ corresponding to two such isomorphisms, V and V'. This pair of isomorphisms may be considered to be close if the basis vectors χ_{mn} and χ'_{mn} are sufficiently close, for all m and n, or, equivalently, if the unitary map $U = V'^*V$, which satisfies $U\chi_{mn} = \chi'_{mn}$, is sufficiently close to the identity.

Let $\Psi \in \mathcal{H}$ with $||\Psi|| = 1$, and let $(|\Psi\rangle\langle\Psi|)_s$ and $(|\Psi\rangle\langle\Psi|)_{s'}$ denote the reduced density operators with co-ordinates defined using (4.6) by

$$\langle\varphi_m|(|\Psi\rangle\langle\Psi|)_s|\varphi_{m'}\rangle = \sum_n \langle\chi_{mn}|\Psi\rangle\langle\Psi|\chi_{m'n}\rangle,$$

$$\langle\varphi_m|(|\Psi\rangle\langle\Psi|)_{s'}|\varphi_{m'}\rangle = \sum_n \langle\chi'_{mn}|\Psi\rangle\langle\Psi|\chi'_{m'n}\rangle.$$

It is clear that, at the level of co-ordinates, $(|\Psi\rangle\langle\Psi|)_{s'} = (U^*|\Psi\rangle\langle\Psi|U)_s$.

In order to make comparisons between different subsystems we lift structure to the total space $\mathcal{H}$. Thus, given an eigenvector ξ of $(|\Psi\rangle\langle\Psi|)_s$ and an eigenvector ξ' of $(|\Psi\rangle\langle\Psi|)_{s'}$, we compare the corresponding lifted projections $V^*(|\xi\rangle\langle\xi| \otimes 1)V$ and $V'^*(|\xi'\rangle\langle\xi'| \otimes 1)V'$, or, equivalently, we compare $V^*(|\xi\rangle\langle\xi| \otimes 1)V$ and $UV^*(|\xi'\rangle\langle\xi'| \otimes 1)VU^*$. Notice that if U is close to the identity, then $V^*((|\Psi\rangle\langle\Psi|)_s \otimes 1)V$ is certainly close to $V'^*((|\Psi\rangle\langle\Psi|)_{s'} \otimes 1)V'$.

example 7.3 *Choose $\delta > 0$. There exists a Hilbert space $\mathcal{H}$ which can be expressed as a tensor product $\mathcal{H} = \mathcal{H}_s \otimes \mathcal{H}_e$ in two possible ways, corresponding to bases $(\chi_{mn})_{m=1}^{N_s}{}_{n=1}^{N_e}$ and $(\chi'_{mn})_{m=1}^{N_s}{}_{n=1}^{N_e}$ which are related by a unitary transformation U and are close in the sense that $||U - 1|| < \delta$. There is a vector $\Psi \in \mathcal{H}$ such that for any pair (ξ, ξ') consisting of an eigenvector ξ of $(|\Psi\rangle\langle\Psi|)_s$ and an eigenvector ξ' of $(|\Psi\rangle\langle\Psi|)_{s'}$ we have that*

$$\mathrm{tr}((V^*(|\xi\rangle\langle\xi| \otimes 1)V - V'^*(|\xi'\rangle\langle\xi'| \otimes 1)V')^2) > 1.$$

proof Choose $\mathcal{H} = \mathbb{C}^4$. For ε sufficiently small, there is a unitary map U_ε arbitrarily close to the identity acting on $\mathcal{H}$ such that $U_\varepsilon \Psi_{\sigma_\varepsilon} = \Psi_{\rho_\varepsilon}$. Let $\Psi = \Psi_{\rho_\varepsilon} = \sqrt{\frac{1}{2} + \varepsilon} f_{11} + \sqrt{\frac{1}{2} - \varepsilon} f_{22}$, where $(f_{mn})_{m=1}^{2}{}_{n=1}^{2}$ is a basis for $\mathbb{C}^4$. Define $V : \mathcal{H} \to \mathbb{C}^2 \otimes \mathbb{C}^2$ by $V f_{mn} = e_m \otimes e_n$, where $(e_m)_{m=1}^{2}$ is a basis for $\mathbb{C}^2$, and let $V' = V U_\varepsilon^*$.

$$\begin{aligned}\Psi = U_\varepsilon \Psi_{\sigma_\varepsilon} &= \tfrac{1}{2}\left(\sqrt{\tfrac{1}{2}+\varepsilon} + \sqrt{\tfrac{1}{2}-\varepsilon}\,\right) U_\varepsilon f_1 + \tfrac{1}{2}\left(\sqrt{\tfrac{1}{2}+\varepsilon} - \sqrt{\tfrac{1}{2}-\varepsilon}\,\right) U_\varepsilon f_2 \\ &+ \tfrac{1}{2}\left(\sqrt{\tfrac{1}{2}+\varepsilon} - \sqrt{\tfrac{1}{2}-\varepsilon}\,\right) U_\varepsilon f_3 + \tfrac{1}{2}\left(\sqrt{\tfrac{1}{2}+\varepsilon} + \sqrt{\tfrac{1}{2}-\varepsilon}\,\right) U_\varepsilon f_4.\end{aligned}$$

For ε sufficiently small, as U_ε is arbitrarily close to the identity, $\mathrm{tr}((V^*(|\xi\rangle\langle\xi| \otimes 1)V - V'^*(|\xi'\rangle\langle\xi'| \otimes 1)V')^2)$ is arbitrarily close to $\mathrm{tr}((|\xi\rangle\langle\xi| \otimes 1 - |\xi'\rangle\langle\xi'| \otimes 1)^2)$, which is equal to 2 for any pair of eigenvectors of $(|\Psi\rangle\langle\Psi|)_s = (|\Psi_{\rho_\varepsilon}\rangle\langle\Psi_{\rho_\varepsilon}|)_s$ and $(|\Psi\rangle\langle\Psi|)_{s'} = (|\Psi_{\sigma_\varepsilon}\rangle\langle\Psi_{\sigma_\varepsilon}|)_s$. ∎

8 Conclusion.

The results contained in this paper can be seen as clarifying certain features of the modal interpretation of quantum mechanics, and thus as contributions towards an assessment of the merits of that interpretation, at least in the versions considered here. The paper contains both positive and negative results.

We have discussed theorems about continuity and analyticity of eigenvalues and eigenvectors of reduced density operators. We have shown that, under certain conditions, it is possible to establish the continuity of eigenvectors even at degeneracy points. We have used these continuity properties to introduce the notion of assignable projections for a reduced state. We have formulated a version of the modal interpretation which attributes properties in terms of assignable projections, extending the conventional modal interpretation which uses the eigenprojections in the spectral resolution of the reduced state. The properties attributed in our extended modal interpretation can be described in terms of continuous trajectories. This constitutes a major advantage over the conventional modal interpretation, and also provides a framework for the derivation of the dynamics of the actually possessed properties.

On the other hand, the physical adequacy of the modal interpretation must be called into question. We have shown that the continuous trajectories of the definite properties exhibit instabilities in the neighbourhood of a degeneracy point. Variations affecting only slightly the reduced state can induce radical changes in the definite properties. These radical changes may be induced using arbitrarily small energies, or by an arbitrarily slight misidentification of the system considered.

The properties attributed in the modal interpretation can also fail to be physically adequate in the sense that they may not reflect the predicted behaviour of a system. A similar point is made by Albert and Loewer [9, 10], who concentrate on a specific model of measurements. In example 5.1, we have seen that eigenvectors can disappear when the evolution of the system state is periodic. Here are two further examples:

example 8.1 *For $t \geq 0$, $\lambda > 0$, let*

$$\rho(t) = \tfrac{1}{2}\begin{pmatrix} 1 + e^{-\lambda t}\cos 2\theta t & e^{-\lambda t}\sin 2\theta t \\ e^{-\lambda t}\sin 2\theta t & 1 - e^{-\lambda t}\cos 2\theta t \end{pmatrix}.$$

$\rho(t)$ has eigenvectors $\begin{pmatrix} \cos\theta t \\ \sin\theta t \end{pmatrix}$ *and* $\begin{pmatrix} \sin\theta t \\ -\cos\theta t \end{pmatrix}$.

In this example, the density matrix is approaching equilibrium, but the eigenvectors do not reflect this approach.

example 8.2 *Let $H = -\dfrac{d^2}{dx^2}$ on $L^2[0,1]$ with boundary conditions $\psi(0) = \psi(1) = 0$. Let $\rho = e^{-H}/\operatorname{tr}(e^{-H})$. ρ is non-degenerate and has unique eigenvector expansion*

$$\rho = \sum_{n=1}^{\infty} e^{-\pi^2 n^2} |u_n\rangle\langle u_n| / Z,$$

where $u_n(x) = \sqrt{2}\sin n\pi x$.

This is a model for an individual particle in an ideal gas in one dimension. The eigenvectors are entirely delocalized states, so that, in this situation, the properties which are definite according to the modal interpretation are no less "quantum mechanical" than the original state. One might wish to claim that in a real (non-ideal) gas in three dimensions with many particles, the eigenvectors would be, or using degeneracy could be chosen to be, wavefunctions for localized particles. However, it seems implausible that this is always necessarily true. It may very well be the case that the density operator for such a situation is close to a density operator with that type of eigenvector, but, as we have seen repeatedly in this paper, close density operators do not necessarily have close eigenvectors.

Finally, it is perhaps worth remarking that even the most preliminary supposition of the modal interpretation can be called into question. This is the supposition that a quantum mechanical system S can be defined on a Hilbert space $\mathcal{H}_s$ with its environment defined on $\mathcal{H}_e$ and that the total Hilbert space of the universe takes the form $\mathcal{H} = \mathcal{H}_s \otimes \mathcal{H}_e$. In relativistic quantum field theory [21], the natural subsystems to work with are those associated with subsets of space-time. Such subsystems are defined not by "Type I" von Neumann algebras, like the set of all bounded operators on $\mathcal{H}_s$, but by "Type III" von Neumann algebras. It is still possible to define the reduction (ρ) to such an algebra of the univeral state ($|\Psi\rangle\langle\Psi|$), but no eigenvector decomposition of ρ is now possible. The examples in this paper demonstrating that eigenvector decompositions may not behave well under approximations suggest that the modal interpretation is not at liberty to ignore this difficulty by approximating the speed of light to infinity.

Acknowledgements We would like to thank Klaas Landsman for discussions. GB acknowledges support from the Arnold Gerstenberg Fund; PV acknowledges support from the Netherlands Organisation for Scientific Research (NWO) and acknowledges the Amsterdam office of the British Council for offering a generous Fellowship to visit the Department of History and Philosophy of Science at the University of Cambridge.

References

[1] Kochen, S., 'A New Interpretation of Quantum Mechanics', in P. Lahti and P. Mittelstaedt (eds), *Symposium on the Foundations of Modern Physics* (Singapore: World Scientific, 1985), pp. 151–169.

[2] Krips, H., *The Metaphysics of Quantum Theory* (Oxford: Clarendon Press, 1987).

[3] Dieks, D., 'The Formalism of Quantum Theory: an Objective Description of Reality?', *Ann. der Phys.* **7** (1988), 174–190.

[4] Healey, R., *The Philosophy of Quantum Mechanics: An Interactive Interpretation* (Cambridge: Cambridge University Press, 1989).

[5] Vermaas, P.E. and Dieks, D., 'The Modal Interpretation of Quantum Mechanics and Its Generalization to Density Operators', *Found. Phys.* **25** (1995), 145–158.

[6] Clifton, R., 'Independently Motivating the Kochen–Dieks Modal Interpretation of Quantum Mechanics', *Brit. J. Phil. Sci.* **46** (1995), 33–57.

[7] Bacciagaluppi, G. and Hemmo, M., 'Modal Interpretations, Decoherence and Measurements', forthcoming in *Stud. Hist. Phil. Mod. Phys.* **27:2** (June 1996).

[8] Donald, M.J., 'A Mathematical Characterization of the Physical Structure of Observers', *Found. Phys.* **25** (1995), 529–571.

[9] Albert, D. and Loewer, B., 'Wanted Dead or Alive: Two Attempts to Solve Schrödinger's Paradox', in A. Fine, M. Forbes and L. Wessels (eds), *Proceedings of the 1990 Biennial Meeting of the Philosophy of Science Association*, Vol. 1 (East Lansing: Philosophy of Science Association, 1990), pp. 277–285.

[10] Albert, D. and Loewer, B., 'Non-Ideal Measurements', *Found. Phys. Lett.* **6** (1993), 297–305.

[11] Rellich, F., *Perturbation Theory of Eigenvalue Problems* (New York: Gordon and Breach, 1969).

[12] Kato, T., *Perturbation Theory for Linear Operators* (Berlin: Springer, 1966. 2nd edition, reprinted 1995).

[13] Reed, M. and Simon, B., *Methods of Modern Mathematical Physics*, Vol. 4 (New York: Academic Press, 1978).

[14] Reed, M. and Simon, B., *Methods of Modern Mathematical Physics*, Vol. 1 (New York: Academic Press, 1972).

[15] Dieudonné, J., *Foundations of Modern Analysis* (New York: Academic Press, 1969).

[16] Simon, B., 'Convergence Theorems for Entropy', *J. Math. Phys.* **14** (1973), 1940–1941.

[17] Reed, M. and Simon, B., *Methods of Modern Mathematical Physics*, Vol. 2 (New York: Academic Press, 1975).

[18] Dieks, D., 'Physical Motivation of the Modal Interpretation of Quantum Mechanics', *Phys. Lett.* **A 197** (1995), 367–371.

[19] Vermaas, P.E., 'Unique Transition Probabilities in the Modal Interpretation', forthcoming in *Stud. Hist. Phil. Mod. Phys.* **26:3** (December 1995).

[20] Bacciagaluppi, G. and Dickson, M., 'Modal Interpretations with Dynamics', in preparation.

[21] Haag, R., *Local Quantum Physics: Fields, Particles, Algebras* (Berlin: Springer, 1992).