Part IA NT Maths Tripos
Model Answer– Paper II 2002 1A
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Comments

I have been an examiner for part 1A NST Mathematics on several occasions. Marking this examination is quite hard work. There are usually six examiners who set and mark both papers, examiners marking their own questions. On average each examiner marks about 1000 attempts. Marking is quite hard work because most students make mistakes and (contrary to what you might think) we try to give marks for nuggets of sensible working embedded in every attempt. In addition the time available for marking is very short, just a few days.

In 2002 I was asked to mark examiner A's questions at the last moment because that examiner was suddenly unavailable. Thus I found myself having to make about 450 attempts a question 1A at very short notice and without a model answer. (With hindsight I guess I could have asked for a model answer, but at the time I didn't think that would be necessary).

I soon discovered that part (b) of this question was harder then I expected and that the candidates were also unable to get a sensible answer.

Basically part (b) is simply solving a linear vector equation for the unknown vector $\mathbf{r} = (x, y, z)$. This is equivalent to solving three simultaneous linear equations for (x, y, z). I used Mathematica to do this (unfortunately not possible in examinations!) and then re-expressed the answer in vector notation. Seeing the form of the answer made it easy to find the solution given below.

Out of the 450 attempts there was only one single correct solution and that student had use the same brute force method of solving three complicated simultaneous equation. The candidate actually did this by explicitly inverting a horrible 3×3 matrix. Needless to say he got full marks. Quite a lot of students tried either a dot product or a cross product – but nobody tried both. The majority of candidates ignored algebraic methods entirely and resorted to complicated geometric and usually vague arguments, these answers were very difficult to mark!

After this experience I added this question to the 1A handout questions and discussed it as an example in my lectures – so you may well have seen the solution already.

Question:

(a) if $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are non-coplanar show that $\mathbf{a}, \mathbf{a}\times\mathbf{b}$ and $\mathbf{a}\times\mathbf{c}$ are non-coplanar [8]

(b) solve the vector equation $\mathbf{a}\times\mathbf{r} + \lambda\mathbf{r} = \mathbf{c}$ for $\mathbf{r}$, where $\lambda \neq 0$. [12]

Solution:

Part (a) - A geometric answer is no doubt possible here, but an algebraic answer is clearer and probably preferred by examiners. (Certainly preferred in my case).

Recall that $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are non-coplanar if and only if (iff) $[\mathbf{a},\mathbf{b},\mathbf{c}] \neq 0$ where the scalar triple product is cyclic in $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ and is defined as $\mathbf{a}.(\mathbf{b}\times\mathbf{c})$ which is the volume of the 3D parallelepiped defined by the three vector and is actually the determinant of their components.

So lets look at the triple scalar product of $\mathbf{a}, \mathbf{a}\times\mathbf{b}$ and $\mathbf{a}\times\mathbf{c}$; and for simplicity considerer:

$$\begin{aligned}[(\mathbf{a}\times\mathbf{c}),\mathbf{a},(\mathbf{a}\times\mathbf{b})] &= (\mathbf{a}\times\mathbf{c})\cdot(\mathbf{a}\times(\mathbf{a}\times\mathbf{b})) \\ &= (\mathbf{a}\times\mathbf{c})\cdot((\mathbf{a}\cdot\mathbf{b})\mathbf{a}-(\mathbf{a}\cdot\mathbf{a})\mathbf{b}) \\ &= a^2(\mathbf{a}\times\mathbf{c})\cdot\mathbf{b} \\ &= -a^2[\mathbf{a},\mathbf{b},\mathbf{c}]\end{aligned}$$

Note that we have used the standard formula for a repeated vector product in line 2. This result is clearly non zero if $[\mathbf{a},\mathbf{b},\mathbf{c}] \neq 0$ so the vectors are non-coplanar as required. Many students did well on this part.

Part (b)

Find $\mathbf{r}$ given the vector equation:

$$\mathbf{a}\times\mathbf{r}+\lambda\mathbf{r}=\mathbf{c} \qquad \text{Eqn 1}$$

Take dot product of $\mathbf{a}$ and Eqn 1:

$$\mathbf{a}\cdot[\text{Eqn 1}] \quad \Rightarrow \quad \mathbf{a}.(\mathbf{a}\times\mathbf{r})+\lambda\mathbf{a}\cdot\mathbf{r}=\mathbf{a}\cdot\mathbf{c} \quad \Rightarrow \quad \mathbf{a}\cdot\mathbf{r}=\frac{\mathbf{a}\cdot\mathbf{c}}{\lambda} \qquad \text{Eqn 2}$$

This looks good, but Eqn 2 is a single scalar equation and thus insufficient on its own to determine the 3-component vector $\mathbf{r}$.

Now also take vector product of $\mathbf{a}$ and Eqn 1:

$$\mathbf{a}\times[\text{Eqn 1}] \quad \Rightarrow \quad \mathbf{a}\times(\mathbf{a}\times\mathbf{r})+\lambda\mathbf{a}\times\mathbf{r}=\mathbf{a}\times\mathbf{c} \qquad \text{Eqn 3}$$

Expand the repeated vector product:

$$(\mathbf{a}\cdot\mathbf{r})\mathbf{a}-(\mathbf{a}\cdot\mathbf{a})\mathbf{r}+\lambda\,\mathbf{a}\times\mathbf{r}=\mathbf{a}\times\mathbf{c}$$

Hence using eqn 2 to replace $\mathbf{a}\cdot\mathbf{r}$ in the first term and Eqn 1 to replace $\mathbf{a}\times\mathbf{r}$ in the third term we get:

$$\frac{\mathbf{a}\cdot\mathbf{c}}{\lambda}-a^2\mathbf{r}+\lambda(\mathbf{c}-\lambda\mathbf{r})=\mathbf{a}\times\mathbf{c} \qquad \text{Eqn 4}$$

and since eqn 4 just involves $\mathbf{r}$ multiplied by scalars we can rearrange to get:

$$-a^2\mathbf{r}-\lambda^2\mathbf{r}=\mathbf{a}\times\mathbf{c}-\frac{\mathbf{a}\cdot\mathbf{c}}{\lambda}\mathbf{a}-\lambda\mathbf{c} \quad \text{or:}$$

$$\boxed{\mathbf{r}=\frac{\lambda^2\mathbf{c}+(\mathbf{a}\cdot\mathbf{c})\mathbf{a}-\lambda(\mathbf{a}\times\mathbf{c})}{\lambda(a^2+\lambda^2)}}$$

Further comments:

The final result is quite ugly and has no particular geometrical interpretation; this is why all the complicated geometrical arguments I had to mark in the examination were doomed to fail. The answer it is in fact just the solution to three complicated linear simultaneous equations for (x, y, z) written in vector form.

This is not a general solution to the problem:

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} p \\ q \\ r \end{bmatrix} \qquad \text{Eqn 5}$$

since this has up to 12 arbitrary constants, whereas Eqn 1 has only 7 constants (λ and the components of $\mathbf{a}$ and $\mathbf{c}$). Nevertheless Eqn 1 can be written in the form of Eqn 5 and hence a brute force solution can be found by inverting the resulting matrix:

$$\begin{bmatrix} \lambda & -a_z & a_y \\ a_z & \lambda & -a_x \\ -a_y & a_x & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} c_x \\ c_y \\ c_z \end{bmatrix} \qquad \text{Eqn 1*}$$

Remarkably inverting this matrix is exactly what one (and only one) student managed to do in the examination.