Part IA NT Maths Tripos
Model Answer– Paper II 2008 Y12
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Find: $I(a,b) = \int_{-\infty}^{\infty} e^{-\frac{1}{2}(ax^2+b/x^2)} dx$,

given $\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} dx = \sqrt{2\pi}$.

and the hint to try the substitution $y = \sqrt{a}x - \sqrt{b}/x$, (we have omitted an extra factor $\frac{1}{2}$ in the original question, as it is slightly unhelpful).

This question is unreasonably hard!

Note $I(a,b) = 2\int_{0}^{\infty} e^{-\frac{1}{2}(ax^2+b/x^2)} dx$ by symmetry, and y goes from $-\infty$ to ∞ as x goes from 0 to ∞. Also $y^2 = ax^2 + b/x^2 - 2\sqrt{ab}$, or $ax^2 + b/x^2 = y^2 + 2\sqrt{ab}$.

Thus using the substitution we get:

$$I(a,b) = 2\int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2-\sqrt{ab}} dx \qquad \text{Eqn 1}$$

Unfortunately

$$dy = (\sqrt{a} + \sqrt{b}/x^2)\, dx . \qquad \text{Eqn 2}$$

This makes it hard to continue; there are probably several ways of getting the integral done, we will follow Richard Feynman's lead[#] and differentiate $I(a,b)$ w.r.t. the parameter b :

$$\begin{aligned}\frac{\partial I}{\partial b} &= 2\int_{0}^{\infty} -\frac{1}{2x^2} e^{-\frac{1}{2}(ax^2+b/x^2)} dx \\ &= -\int_{0}^{\infty} \frac{1}{x^2} e^{-\frac{1}{2}y^2-\sqrt{ab}} dx.\end{aligned}$$

So we can write:

$$\begin{aligned}\sqrt{a}I - 2\sqrt{b}\frac{\partial I}{\partial b} &= 2\int_{0}^{\infty} e^{-\frac{1}{2}y^2-\sqrt{ab}} (\sqrt{a} + \sqrt{b}/x^2)\, dx, \\ &= 2e^{-\sqrt{ab}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2} dy \\ &= 2e^{-\sqrt{ab}} \sqrt{2\pi}.\end{aligned}$$

[#] "Surely You Must Be Joking Mr Feynman", pages 86-87. This book is great fun and a must read for all intending Physicists.

Rearranging we get a standard first order differential equation for I:

$$\frac{\partial I}{\partial b} - \frac{1}{2}\sqrt{\frac{a}{b}}I = -\sqrt{\frac{2\pi}{b}}e^{-\sqrt{ab}}$$

We can use an integrating factor $\mu(b) = e^{\int -\frac{1}{2}\sqrt{\frac{a}{b}}db} = e^{-\sqrt{ab}}$ to get:

$$\frac{\partial}{\partial b}(Ie^{-\sqrt{ab}}) = -\sqrt{2\pi}\,\frac{1}{b}e^{-2\sqrt{ab}},$$

And finally $I = \sqrt{\frac{2\pi}{a}}e^{-\sqrt{ab}} + K(a)$ where *K(a)* is an arbitrary function of *a*, but noting the value of the integral for $b = 0$ we set $K = 0$ giving the final answer (for 15 marks):

$$I(a,b) = \sqrt{\frac{2\pi}{a}}e^{-\sqrt{ab}}$$

The last part can be easily done by using the obvious substitution $y = 1/x$ or by noting we have already found this integral is $-\frac{\partial I}{\partial b}$. [Hence the answer is $I(b,a)$].

The above is my solution found in May 2012.

Or (October 2014) we can deal directly with Eqn 1 as follows:

From $y = \sqrt{a}x - \sqrt{b}/x$ we have the quadratic equation $\sqrt{a}x^2 - xy - \sqrt{b} = 0$ thus

$$x = \frac{y \pm \sqrt{y^2 - 4\sqrt{ab}}}{2\sqrt{a}} \quad \text{so} \quad dx = \frac{dy}{2\sqrt{a}} \pm \frac{ydy}{\sqrt{y^2 - 4\sqrt{ab}}}.$$

Substituting for dx in Eqn 1:

$$I(a,b) = 2e^{-\sqrt{ab}}\left[\int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2}\,\frac{dy}{2\sqrt{a}} \pm \int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2}\,\frac{y\,dy}{\sqrt{y^2 - 4\sqrt{ab}}}\right]$$

This looks horrible but the second term with $y\,dy$ is odd thus the second nasty integral vanishes! The first term is just a multiple of the standard Gaussian integral, so:

$$I(a,b) = \sqrt{\frac{2\pi}{a}}e^{-\sqrt{ab}} \quad \text{QED.}$$