Instability, Isolation, and the Tridecompositional Uniqueness Theorem. *

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The tridecompositional uniqueness theorem of Elby and Bub (1994) shows that a wavefunction in a triple tensor product Hilbert space has at most one decomposition into a sum of product wavefunctions with each set of component wavefunctions linearly independent. I demonstrate that, in many circumstances, the unique component wavefunctions and the coefficients in the expansion are both hopelessly unstable, both under small changes in global wavefunction and under small changes in global tensor product structure. In my opinion, this means that the theorem cannot underlie law-like solutions to the problems of the interpretation of quantum theory. I also provide examples of circumstances in which there are open sets of wavefunctions containing no states with various decompositions.

1. Introduction.

The central problem of the interpretation of quantum theory is to explain and characterize the existence, or apparent existence, of state "collapse".

State collapse is the process by which, for example, an electron, despite apparently going through a double slit as an extended wave, always appears to make a well-localized impact on a screen at only one of many possible places – the extended wave "collapses" to a localized state. Different interpretations of quantum theory suggest different ways of understanding such collapses. One plausible goal would be to propose the existence of laws of nature defining the circumstances in which collapse occurs or appears to occur. I have attempted this myself based on a characterization of observers and their information (Donald 1999). To propose such laws, whether or not they involve observers, is to propose a realist interpretation of quantum theory, in the sense that the laws are supposed to be truths about reality which are independent of our abilities to verify them.

There are many proposals simpler than my own. Perhaps the simplest is implicit in a not uncommon understanding of decoherence theory. This assumes that, as a consequence of decoherence, a global quantum state just falls naturally into a family of quasi-classical pieces, each of which describes the observation of an individual collapse outcome. The problem with this is that decoherence theory does not by itself solve the preferred basis problem. It merely provides a framework within which a quasi-classical solution to the preferred basis problem is not ruled out. Typical quantum states for macroscopic objects can be split into quasi-classical pieces in many ways which, at least at a level of fine structure, are mutually incompatible. But a realist explanation

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of collapse along these lines requires specification of one fundamental splitting. Which individual piece we see may be a matter of probability, but to define probabilities we need to be given a splitting into a definite set of possible pieces. Although there are models which provide asymptotically unambiguous decompositions if we wait long enough (Hepp 1972) or look on a broad enough scale (Ollivier, Poulin, and Zurek 2003, 2004), such models are not sufficient. In particular, if we try to analyse the detailed real-time functioning of an individual human brain, there is considerable ambiguity as to the precise scales on which information is being experienced (Donald 2002).

Another approach supposes that a natural splitting of a global quantum state might be a consequence of some property of the mathematics of quantum states. The paradigm is the Schmidt, or biorthogonal, decomposition. If the global Hilbert space $\mathcal{H}$ splits naturally into a tensor product $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ and if the global state is a pure state, then its wavefunction Ψ has a decomposition of the form $\Psi = \sum_k \sqrt{p_k}\psi^1_k\psi^2_k$, where $p_k \geq 0$ for all k, $\sum_k p_k = 1$, and (ψ^1_k) and (ψ^2_k) are orthonormal bases for $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Generically, the Schmidt decomposition is unique. It fails to be unique at points of degeneracy, which are precisely those points where the p_k are not all distinct.

The idea of using this decomposition to explain collapse or the appearance of collapse is the idea that Ψ will correspond to an "extended wave" of some form, satisfying a global Schrödinger equation of some form, and the component states $\psi^1_k\psi^2_k$ will correspond to observed collapse states. The great advantage of this idea is that the Schmidt decomposition is well-defined. If a global quantum state is given, then, generically, its Schmidt decomposition is determined. Good mathematical definitions are the required underpinnings for realist laws of nature. A law stating that Ψ will collapse or appear to collapse to component $\psi^1_k\psi^2_k$ with probability p_k is at the heart of the modal interpretation (or some versions of it). Bub (1997) provides a brief review and Vermaas (2000) a detailed examination.

A well-defined law faces questions. Among the questions which arise for the Schmidt decomposition are:

1.1) How are the subsystems $\mathcal{H}_1$ and $\mathcal{H}_2$ to be identified?

1.2) Is it appropriate to assume that the state on $\mathcal{H}_1 \otimes \mathcal{H}_2$ is pure?

1.3) Given plausible global wavefunctions Ψ on $\mathcal{H}_1 \otimes \mathcal{H}_2$, are the component wavefunctions $\psi^1_k\psi^2_k$ appropriately quasi-classical?

1.4) What about degeneracy points?

In my opinion, the modal interpretation fails because none of these questions can be given entirely satisfactory answers. The literature is extensive and I shall not review it here. For the moment, I merely note that, for macroscopic systems, the answer to question 1.3 seems to be that the component wavefunctions can be quite arbitrary and can be unstable under small changes in Ψ and under small changes in the tensor product structure (Bacciagaluppi, Donald, and Vermaas 1995, Donald 1998).

The tridecompositional uniqueness theorem, stated in various versions in section 2, was developed as a tool to solve problems with the Schmidt decomposition. It has been invoked to explain the states adopted, or apparently adopted, by quantum systems in contact with both a measuring device and an environment (Bub 1997, Schlosshauer 2003). The purpose of this paper is to show that it has its own problems. Instability problems will be exemplified in section 3, culminating in theorems 3.4 and 3.6. In section 4, it will be shown that there are open sets, and other significant sets, which contain no wavefunctions with various decompositions. In section 5, in a counterpoint to the central thrust of the paper, detailed technical estimates will be used to show that triorthogonal decompositions – those satisfying theorem 2.3 – are in fact stable. The results can be summarized by saying that, in large spaces, general tridecompositions tend to exist but are unstable, while triorthogonal decompositions are stable but unlikely.

For brevity, it will be assumed throughout that all named Hilbert spaces are non-trivial, and in particular that they have dimension at least two.

2. The Tridecompositional Uniqueness Theorem.

Theorem 2.1 *Let $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$ be a triple tensor product of Hilbert spaces and let $\Psi \in \mathcal{H}$ be a wavefunction.*

Then Ψ has at most one decomposition of the form $\Psi = \sum_{k=1}^{K} a_k \psi_k^1 \psi_k^2 \psi_k^3$ where K is finite, $|a_k| > 0$ for $k = 1, \dots, K$, $\{\psi_k^1 : k = 1, \dots, K\} \subset \mathcal{H}_1$ and $\{\psi_k^2 : k = 1, \dots, K\} \subset \mathcal{H}_2$ are linearly independent sets of wavefunctions, and $\{\psi_k^3 : k = 1, \dots, K\} \subset \mathcal{H}_3$ is a set of wavefunctions in $\mathcal{H}_3$ such that no pair is collinear.

Note that $\psi_k^1 \psi_k^2 \psi_k^3$ abbreviates $\psi_k^1 \otimes \psi_k^2 \otimes \psi_k^3$, that a wavefunction ψ is a normalized vector ($||\psi|| = 1$), that a finite set $\{\psi_k : k = 1, \dots, K\}$ of wavefunctions is linearly independent iff $\sum_{k=1}^{K} c_k \psi_k = 0$ implies $c_k = 0$ for all k, and that a pair $\{\psi_k, \psi_l\}$ of wavefunctions is collinear, iff it is not linearly independent, iff ψ_k and ψ_l differ at most by a phase factor. The uniqueness of the decomposition in this theorem, of course, allows for re-orderings of the terms and changes in phase factors.

Theorem 2.1 is proved in Elby and Bub (1994), with the argument improved and completed in Clifton (1994) and Bub (1997). Kirkpatrick (2001) extends the result by noting that it is not necessary to specify which particular pair of spaces have linearly independent wavefunctions. Cassam-Chenaï and Patras (2004) set the theorem in a powerful algebraic framework allowing them to consider spaces of indistinguishable particles. In section 3, I shall show that, in many circumstances, although the decomposition in theorem 2.1 is unique at individual wavefunctions, it can vary wildly as we move from wavefunction to wavefunction. The fundamental source of this instability is that linear independence is a very weak property. If there are enough spare dimensions available, then arbitrarily small modifications can turn a finite set of wavefunctions into a linear independent set.

It will be useful to be able to refer to the statements of the following consequences of theorem 2.1. Elby and Bub have named theorem 2.3 the "triorthogonal uniqueness theorem". We shall refer to a decomposition satisfying theorem 2.3 as a "triorthogonal decomposition", and to a wavefunction which has a triorthogonal decomposition as a "triorthogonal wavefunction".

Theorem 2.2 *Let $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$ be a triple tensor product of Hilbert spaces and let $\Psi \in \mathcal{H}$ be a wavefunction.*

Then Ψ has at most one decomposition of the form $\Psi = \sum_{k=1}^{K} a_k \psi_k^1 \psi_k^2 \psi_k^3$ where K is finite, $|a_k| > 0$ for $k = 1, \ldots, K$, and, for $i = 1, 2, 3$, $\{\psi_k^i : k = 1, \ldots, K\}$ is a linearly independent set of wavefunctions in $\mathcal{H}_i$.

Theorem 2.3 *Let $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$ be a triple tensor product of Hilbert spaces and let $\Psi \in \mathcal{H}$ be a wavefunction.*

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In the statements given here, theorem 2.1 and 2.2 refer only to finite decompositions. There are two reasons for this. Firstly the conventional definition of linear independence states that an infinite set of wavefunctions is linearly independent if and only if every finite subset is linearly independent. For infinite sets $\{\psi_k\}$, this is weaker than the statement that $\sum_k c_k \psi_k = 0$ implies $c_k = 0$. Secondly, the proofs of the theorems involve expanding one linearly independent set in terms of another. For infinite sets, the coefficients of these expansions can, in general, become unbounded.

These problems are not relevant in the case of theorem 2.3. In that case, the decomposition also amounts to a Schmidt decomposition of Ψ in the tensor product of $\mathcal{H}_1$ with $\mathcal{H}_2 \otimes \mathcal{H}_3$. This uniquely identifies the $|a_k|$ and, for each $\delta > 0$, the finite dimensional space spanned by the $\psi_k^1 \psi_k^2 \psi_k^3$ with $|a_k| > \delta$. Theorem 2.3 then follows from the uniqueness of decomposition on these finite spaces, which is a consequence of theorem 2.1.

3. Instability.

Example 3.1 Let $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$ where, for $i = 1, 2, 3$, $(\psi_n^i)_{n=1}^{N_i}$ is an orthonormal basis for $\mathcal{H}_i$. Suppose that $\Psi = \frac{1}{\sqrt{2}}(\psi_1^1 \psi_1^2 \psi_2^3 - \psi_1^1 \psi_2^2 \psi_1^3)$.

Ψ is a tensor product of ψ_1^1 with a singlet wavefunction $\frac{1}{\sqrt{2}}(\psi_1^2 \psi_2^3 - \psi_2^2 \psi_1^3) \in \mathcal{H}_2 \otimes \mathcal{H}_3$.

One of the original aims of Elby and Bub (1994) in developing the tridecompositional uniqueness theorem was to avoid problems which are raised by the non-unique Schmidt decomposition of singlets. This same non-uniqueness, however, can be used to provide an example of instability in the wavefunctions.

Write $\varphi_1^2 = \frac{1}{\sqrt{2}}(\psi_1^2 - \psi_2^2)$ and $\varphi_2^2 = \frac{1}{\sqrt{2}}(\psi_1^2 + \psi_2^2)$ and write $\varphi_1^3 = \frac{1}{\sqrt{2}}(\psi_1^3 + \psi_2^3)$ and $\varphi_2^3 = \frac{1}{\sqrt{2}}(\psi_1^3 - \psi_2^3)$.

Then

$$\Psi = \frac{1}{\sqrt{2}}(\psi_1^1 \varphi_1^2 \varphi_1^3 - \psi_1^1 \varphi_2^2 \varphi_2^3).$$

Let $\varphi_1^1 = \xi_1^1 = \psi_1^1$ and $\varphi_2^1(\theta) = \xi_2^1(\theta) = -\cos\theta \psi_1^1 - \sin\theta \psi_2^1$. Note that $\{\psi_1^1, -\cos\theta \psi_1^1 - \sin\theta \psi_2^1\}$ is linearly independent as long as $\sin\theta \neq 0$.

Write $\xi_1^2 = \psi_1^2$, $\xi_2^2 = \psi_2^2$, $\xi_1^3 = \psi_2^3$, and $\xi_2^3 = \psi_1^3$.

For $0 < \theta \leq \frac{\pi}{2}$,

$$\Phi(\theta) = \frac{1}{\sqrt{2}}(\varphi_1^1\varphi_1^2\varphi_1^3 + \varphi_2^1(\theta)\varphi_2^2\varphi_2^3)$$

and

$$\Psi(\theta) = \frac{1}{\sqrt{2}}(\xi_1^1\xi_1^2\xi_1^3 + \xi_2^1(\theta)\xi_2^2\xi_2^3)$$

both satisfy the conditions of theorem 2.2, and so these expansions are unique and $\Phi(\theta) \neq \Psi(\theta)$. However

$$\lim_{\theta\to 0} \Phi(\theta) = \lim_{\theta\to 0} \Psi(\theta) = \Psi$$

and so the components of the terms in the expansion are not stable. ∎

The use of a singlet is not critical in this example. Indeed, any wavefunction ψ^{23} on $\mathcal{H}_2 \otimes \mathcal{H}_3$ which is not a product can be decomposed into a sum of products with linearly-independent components in many different ways. Such a wavefunction will have a Schmidt decomposition of the form $\psi^{23} = \sum_{k=1}^K \sqrt{p_k}\psi_k^2\psi_k^3$ with $p_1 \geq p_2 > 0$. This can be written in different ways by using the identity

$$\begin{aligned}\sqrt{p_1}\psi_1^2\psi_1^3 + \sqrt{p_2}\psi_2^2\psi_2^3 = (\cos\alpha\psi_1^2 + \sin\alpha\psi_2^2)(\sqrt{p_1}\cos\alpha\psi_1^3 + \sqrt{p_2}\sin\alpha\psi_2^3)\\ +(\sin\alpha\psi_1^2 - \cos\alpha\psi_2^2)(\sqrt{p_1}\sin\alpha\psi_1^3 - \sqrt{p_2}\cos\alpha\psi_2^3).\end{aligned}$$

Instability in tridecompositions close to $\psi_1^1\psi^{23}$ follows as in example 3.1.

Example 3.2 Kirkpatrick (2001) has used theorem 2.1 to prove that a state ρ on a tensor product space $\mathcal{H}_1 \otimes \mathcal{H}_2$ has at most one convex decomposition of the form $\rho = \sum_{k=1}^K w_k|\psi_k^1\psi_k^2><\psi_k^1\psi_k^2|$ where K is finite, $w_k > 0$ for $k = 1,\ldots,K$, and one of the sets $\{\psi_k^1 : k = 1,\ldots,K\} \subset \mathcal{H}_1$ and $\{\psi_k^2 : k = 1,\ldots,K\} \subset \mathcal{H}_2$ is linearly independent while, in the other, no pair of wavefunctions is collinear. Example 3.1 can be extended to show that the components of Kirkpatrick's decomposition are also unstable.

As $\{\varphi_1^3, \varphi_2^3\}$ and $\{\xi_1^3, \xi_2^3\}$ are orthonormal, the partial traces of $|\Phi(\theta)><\Phi(\theta)|$ and $|\Psi(\theta)><\Psi(\theta)|$ on $\mathcal{H}_1 \otimes \mathcal{H}_2$ are given by

$$(|\Phi(\theta)><\Phi(\theta)|)_{12} = \tfrac{1}{2}|\varphi_1^1\varphi_1^2><\varphi_1^1\varphi_1^2| + \tfrac{1}{2}|\varphi_2^1(\theta)\varphi_2^2><\varphi_2^1(\theta)\varphi_2^2|$$

and

$$(|\Psi(\theta)><\Psi(\theta)|)_{12} = \tfrac{1}{2}|\xi_1^1\xi_1^2><\xi_1^1\xi_1^2| + \tfrac{1}{2}|\xi_2^1(\theta)\xi_2^2><\xi_2^1(\theta)\xi_2^2|.$$

For $0 < \theta \leq \frac{\pi}{2}$, these decompositions satisfy the conditions of Kirkpatrick's theorem, so that the components are unique. However, although, as $\theta \to 0$,

$$||(|\Phi(\theta)><\Phi(\theta)|)_{12} - (|\Psi(\theta)><\Psi(\theta)|)_{12}|| \to 0,$$

the components do not converge. Indeed, for all θ, $|<\varphi_i^1\varphi_i^2|\xi_j^1\xi_j^2>| \leq \frac{1}{\sqrt{2}}$ for $i,j \in \{1,2\}$. ∎

Example 3.3 Once again, let $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$ where, for $i = 1,2,3$, $(\psi_n^i)_{n=1}^{N_i}$ is an orthonormal basis for $\mathcal{H}_i$. This time however take $\Psi = \psi_1^1\psi_1^2\psi_1^3$.

Clearly, for any constant X, $\Psi = (1-X)\psi_1^1\psi_1^2\psi_1^3 + X\psi_1^1\psi_1^2\psi_1^3$.

Write $\varphi_1^1 = \psi_1^1$, $\varphi_1^2 = \psi_1^2$, $\varphi_1^3 = \psi_1^3$ and $\varphi_2^1(\theta) = \cos\theta\psi_1^1 + \sin\theta\psi_2^1$, $\varphi_2^2(\theta) = \cos\theta\psi_1^2 + \sin\theta\psi_2^2$, and $\varphi_2^3(\theta) = \cos\theta\psi_1^3 + \sin\theta\psi_2^3$.

For $0 < \theta \leq \frac{\pi}{2}$, let $\Phi(\theta) = (1 - 1/\sqrt{\theta})\varphi_1^1\varphi_1^2\varphi_1^3 + (1/\sqrt{\theta})\varphi_2^1(\theta)\varphi_2^2(\theta)\varphi_2^3(\theta)$ and $\Psi(\theta) = \Phi(\theta)/||\Phi(\theta)||$. $||\Phi(\theta)||$ is bounded and tends to 1 as $\theta \to 0$.

The expansion for $\Psi(\theta)$ satisfies the conditions of theorem 2.2, but, although $\Psi(\theta) \to \Psi$, the coefficients of the expansion diverge as $\theta \to 0$. ∎

Theorem 3.4 *In a triple product of infinite-dimensional spaces, arbitrarily close to every wavefunction are pairs of wavefunctions satisfying the conditions of the tridecompositional theorem with arbitrarily different component wavefunctions.*

Specifically, let $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$ be a triple product of infinite-dimensional spaces and let $\Psi \in \mathcal{H}$ be a wavefunction. Choose $\varepsilon > 0$ and, for $i = 1, 2, 3$, choose an orthonormal basis $(u_n^i)_{n\geq 1}$ for $\mathcal{H}_i$.

Then, there exist wavefunctions Φ_1 and Φ_2 in $\mathcal{H}$ which satisfy the conditions of theorem 2.2 with unique finite expansions

$$\Phi_1 = \sum_{k=1}^{K} a_k \psi_k^1 \psi_k^2 \psi_k^3 \quad \text{and} \quad \Phi_2 = \sum_{m=1}^{M} b_m \varphi_m^1 \varphi_m^2 \varphi_m^3$$

such that $||\Psi - \Phi_1|| < \varepsilon$, $||\Psi - \Phi_2|| < \varepsilon$ and, for $k = 1, \ldots, K$, $m = 1, \ldots, M$, and $i = 1, 2, 3$, there exists $n(k)$ such that $|<\psi_k^i|u_{n(k)}^i>| > 1 - \varepsilon$ while $|<\psi_k^i|\varphi_m^i>| < \varepsilon$.

Thus each component wavefunction of the expansion of Φ_1 is close to some element of a freely chosen basis while being far from every component wavefunction of the expansion of Φ_2.

proof

Let $\mathcal{H}$, Ψ, ε, and $(u_n^i)_{n\geq 1}$ satisfy the hypothesis of the theorem.

Let $P^N = \sum_{n_1=1}^{N} \sum_{n_2=1}^{N} \sum_{n_3=1}^{N} |u_{n_1}^1 u_{n_2}^2 u_{n_3}^3><u_{n_1}^1 u_{n_2}^2 u_{n_3}^3|$.

$P^N \xrightarrow{s} 1$. Choose N_0 such that $N \geq N_0$ implies $||P^N\Psi|| > 0$ and $||\Psi - \Phi^N|| < \varepsilon/2$ where $\Phi^N = P^N\Psi/||P^N\Psi||$.

Lemma Let $(u_n)_{n=1}^N$ be an orthonormal basis for an N-dimensional Hilbert space. Then there exists another orthonormal basis $(v_n)_{n=1}^N$ such that for all pairs k and m, $|<u_k|v_m>| = \frac{1}{\sqrt{N}}$

proof Define $v_m = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} e^{2\pi ikm/N} u_k$ for $m = 1, \ldots, N$.

Then $||v_m||^2 = 1$, and, for $m \neq n$,

$$<v_n|v_m> = \frac{1}{N} \sum_{k=1}^{N} e^{2\pi ik(m-n)/N} = 0.$$

∎

Choose $N \geq N_0$ so that $\frac{1}{\sqrt{N}} < \varepsilon/2$.

For $i = 1, 2, 3$, define $(v_n^i)_{n=1}^N$ in terms of $(u_n^i)_{n=1}^N$ as in the lemma.

Suppose that

$$\Phi^N = \sum_{n_1=1}^{N} \sum_{n_2=1}^{N} \sum_{n_3=1}^{N} a_{n_1,n_2,n_3} u_{n_1}^1 u_{n_2}^2 u_{n_3}^3 = \sum_{n_1=1}^{N} \sum_{n_2=1}^{N} \sum_{n_3=1}^{N} b_{n_1,n_2,n_3} v_{n_1}^1 v_{n_2}^2 v_{n_3}^3 \qquad (3.5)$$

are eigenvector expansions.

Omitting terms with zero coefficients and choosing some ordering for the remaining terms, these expansions can be re-written in the form

$$\Phi^N = \sum_{k=1}^{K} a_k \psi_k^1(0)\psi_k^2(0)\psi_k^3(0) = \sum_{m=1}^{M} b_m \varphi_m^1(0)\varphi_m^2(0)\varphi_m^3(0)$$

where each $\psi_k^i(0) \in \{u_n^i : n = 1, \ldots, N\}$ and each $\varphi_m^i(0) \in \{v_n^i : n = 1, \ldots, N\}$.

These expansions are different, and so they cannot both satisfy the conditions of the tridecompositional theorem. Indeed, in general, neither expansion will, as we will have $K = M = N^3$ and the component wavefunctions will be repeated.

However, as in example 3.1, an arbitrarily slight perturbation in each component is sufficient to produce linear independence.

Let $\psi_k^i(\theta) = \cos\theta\psi_k^i(0) + \sin\theta\, u_{N+1+k}^i$ for $i = 1, 2, 3$ and $k = 1, \ldots, K$ and let $\varphi_m^i(\theta) = \cos\theta\varphi_m^i(0) + \sin\theta\, u_{N+1+m}^i$ for $i = 1, 2, 3$ and $m = 1, \ldots, M$.

Then, for $0 < \theta \leq \frac{\pi}{2}$, $\Phi_1^N(\theta) = \sum_{k=1}^{K} a_k \psi_k^1(\theta)\psi_k^2(\theta)\psi_k^3(\theta)$ and $\Phi_2^N(\theta) = \sum_{m=1}^{M} b_m \varphi_m^1(\theta)\varphi_m^2(\theta)\varphi_m^3(\theta)$ do both satisfy the conditions of theorem 2.2.

Taking θ sufficiently small gives the theorem. ∎

The argument following (3.5) in this proof can be applied to any finite expansion of Φ^N in $P^N\mathcal{H}$. Thus, combining the methods of example 3.3 and theorem 3.4 shows that, in infinite dimensional spaces, both the components and the coefficients defined by the tridecompositional uniqueness theorem are utterly unstable everywhere. This means that the theorem, although it is remarkable and powerful as a mathematical tool, is useless for explanations of the quasi-classical nature of collapse outcomes; in particular as real physical systems have arbitrarily many degrees of freedom, including those involving virtual photons, which can be touched upon.

Theorem 3.4 is concerned with variations in the global wavefunction. It is also important, for the question of the physical relevance of the tridecompositional uniqueness theorem, to recognize that the assumption that a physical Hilbert space has a fundamental tensor product structure may well be incorrect. This recognition can be supported by consideration of the derived and phenomenological nature of localized particles according to relativistic quantum field theory, but, even at a less sophisticated level, it is hard to justify the idea that there are the sort of natural boundaries which would allow the universe to be divided without ambiguity into a system, a measuring apparatus, and an environment.

If the tensor product structure is not a fundamental aspect of reality, then the ultimate laws of nature cannot depend on it. This means that any approximate version of those laws which does depend on a phenomenological assignment of a tensor product structure should be stable under small variations of that assignment. The next theorem will rework theorem 3.4 to show that the decomposition provided by the tridecompositional uniqueness theorem is not stable under such variations.

Let $\mathcal{H}$ be an infinite dimensional Hilbert space. One inelegant but adequate way of defining the structure of a triple product of infinite-dimensional spaces on $\mathcal{H}$ is simply to label any given orthonormal basis in the form $(\Psi_{n_1,n_2,n_3})_{n_1\geq1,n_2\geq1,n_3\geq1}$.

Then $\mathcal{H} \cong \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$ where $\mathcal{H}_1$, for example, corresponds to the space with operators generated by $\sum_{n_2\geq 1, n_3\geq 1} |\Psi_{m_1,n_2,n_3}><\Psi_{n_1,n_2,n_3}|$.

If U is a unitary map on $\mathcal{H}$, then $(U\Psi_{n_1,n_2,n_3})_{n_1\geq 1, n_2\geq 1, n_3\geq 1}$ is an alternative labelled basis and therefore defines an alternative triple product structure. A sufficient, but not a necessary condition, for this second product structure to be close to the first is that U be appropriately close to 1 (the identity map). A strong measure of proximity – the trace class norm ($||\ ||_1$) – will be invoked in the next theorem.

We shall say that a unitary map U on $\mathcal{H}$ moves one tensor product structure $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$ into another $\mathcal{H} = \mathcal{H}'_1 \otimes \mathcal{H}'_2 \otimes \mathcal{H}'_3$ if, for $i = 1, 2, 3$, there are orthonormal bases $(u^i_n)_{n\geq 1}$ (respectively $(\hat{u}^i_n)_{n\geq 1}$) for $\mathcal{H}_i$ (resp. $\mathcal{H}'_i$) such that, for all triples (n_1, n_2, n_3), $Uu^1_{n_1}u^2_{n_2}u^3_{n_3} = \hat{u}^1_{n_1}\hat{u}^2_{n_2}\hat{u}^3_{n_3}$. We shall say that the structures differ in trace norm by the infimum of $||U - 1||_1$ over all such U.

Using such a unitary map, aspects of the different structures can be compared by asking how they would differ if bases moved by U were identified. For example, if $\psi^1 \in \mathcal{H}_1$ and $\hat{\varphi}^1 \in \mathcal{H}'_1$, then $<\psi^1|\hat{\varphi}^1>$ is undefined because $\mathcal{H}_1$ and $\mathcal{H}'_1$ are different spaces but $<\psi^1 u^2_{n_2} u^3_{n_3}|U^*|\hat{\varphi}^1 \hat{u}^2_{n_2}\hat{u}^3_{n_3}>$, which is independent of n_2 and of n_3 and of the choice of bases moved by U, is an appropriate substitute. We shall denote this expression by $<\psi^1|\hat{\varphi}^1>_U$ and will use a similar notation for analogous expressions. In fact, the identification of bases defines isomorphisms between $\mathcal{H}_i$ and $\mathcal{H}'_i$ for $i = 1, 2, 3$.

Theorem 3.6 *Let $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$ be a triple product of infinite-dimensional spaces and let $\Psi \in \mathcal{H}$ be a wavefunction. Choose $\varepsilon > 0$. Then:*

(3.7) There exists a wavefunction $\Phi_1 \in \mathcal{H}$ with $||\Psi - \Phi_1|| < \varepsilon$ which satisfies the conditions of theorem 2.2 with a unique finite expansion $\Phi_1 = \sum_{k=1}^K a_k\psi^1_k\psi^2_k\psi^3_k$.

There also exists another representation $\mathcal{H} = \mathcal{H}'_1 \otimes \mathcal{H}'_2 \otimes \mathcal{H}'_3$ of $\mathcal{H}$ as a triple product of infinite-dimensional spaces, such that:

(3.8) The two triple product structures differ in trace norm by less than 4ε.

(3.9) In the second structure, Φ_1 also satisfies the conditions of theorem 2.2 with a unique finite expansion $\Phi_1 = \sum_{m=1}^M b_m\hat{\varphi}^1_m\hat{\varphi}^2_m\hat{\varphi}^3_m$.

(3.10) For $k = 1, \ldots, K$, $m = 1, \ldots, M$, and $i = 1, 2, 3$, $|<\psi^i_k|\hat{\varphi}^i_m>_U| < \varepsilon$.

proof Adopt the notation and definitions of theorem 3.4. (3.7) follows immediately.

As they have different unique expansions, clearly $\Phi_1 \neq \Phi_2$.

Φ_1 and Φ_2 can be written in the forms $\Phi_1 = \alpha\Phi_2 + \beta\Phi_2^\perp$ and $\Phi_2 = \bar{\alpha}\Phi_1 - \beta\Phi_1^\perp$ where $<\Phi_2^\perp|\Phi_2> = 0$, $<\Phi_1^\perp|\Phi_1> = 0$, and $|\alpha|^2 + |\beta|^2 = 1$. The choice of phases here implies that $<\Phi_1^\perp|\Phi_2^\perp> = <\Phi_2|\Phi_1> = \alpha$ and so

$$||\Phi_1 - \Phi_2||^2 = ||\Phi_1^\perp - \Phi_2^\perp||^2 = 2 - \alpha - \bar{\alpha}.$$

Then a unitary map U on $\mathcal{H}$ mapping Φ_2 to Φ_1 can be defined by

$$U = |\Phi_1><\Phi_2| + |\Phi_1^\perp><\Phi_2^\perp| + 1 - |\Phi_1><\Phi_1| - |\Phi_1^\perp><\Phi_1^\perp|.$$

This is the unitary map which the identity except on the space spanned by Φ_1 and Φ_2 on which it sends Φ_2 to Φ_1 and $\Phi_2^\perp$ to $\Phi_1^\perp$.

On that space $U - 1$ has matrix representation $U - 1 = \begin{pmatrix} \alpha - 1 & -\bar{\beta} \\ \beta & \bar{\alpha} - 1 \end{pmatrix}$.

It is clear that $U \to 1$ as $\Phi_1 \to \Phi_2$. Matrix multiplication shows that $(U-1)^*(U-1) = (2-\alpha-\bar{\alpha})\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ from which it follows that

$$||U-1||_1 = 2\sqrt{2-\alpha-\bar{\alpha}} = 2||\Phi_1 - \Phi_2|| < 4\varepsilon.$$

This is the bound in (3.8).

Move the triple product structure $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$ with U to $\mathcal{H} = \mathcal{H}'_1 \otimes \mathcal{H}'_2 \otimes \mathcal{H}'_3$. For $i = 1, 2, 3$, define bases $(\hat{u}^i_n)_{n\geq 1}$ for $\mathcal{H}'_i$ by $U|u^1_{n_1} u^2_{n_2} u^3_{n_3}\rangle = |\hat{u}^1_{n_1} \hat{u}^2_{n_2} \hat{u}^3_{n_3}\rangle$, and if $\varphi^i_m = \sum_n f^i_{mn} u^i_n$ then define $\hat{\varphi}^i_m = \sum_n f^i_{mn} \hat{u}^i_n$.

Suppose that $\Phi_2 = \sum_{n_1,n_2,n_3\geq 1} c_{n_1,n_2,n_3} u^1_{n_1} u^2_{n_2} u^3_{n_3}$. Then

$$\Phi_1 = U\Phi_2 = \sum_{n_1,n_2,n_3\geq 1} c_{n_1,n_2,n_3} U|u^1_{n_1} u^2_{n_2} u^3_{n_3}\rangle = \sum_{m=1}^{M} b_m U|\varphi^1_m \varphi^2_m \varphi^3_m\rangle$$

$$= \sum_{m=1}^{M} b_m |\hat{\varphi}^1_m \hat{\varphi}^2_m \hat{\varphi}^3_m\rangle$$

and

$$\langle\psi^i_k|\hat{\varphi}^i_m\rangle_U = \langle\psi^i_k u^2_{n_2} u^3_{n_3}|U^*|\hat{\varphi}^i_m \hat{u}^2_{n_2} \hat{u}^3_{n_3}\rangle = \langle\psi^i_k u^2_{n_2} u^3_{n_3}|\varphi^i_m u^2_{n_2} u^3_{n_3}\rangle = \langle\psi^i_k|\varphi^i_m\rangle.$$

Thus Φ_1 has the expansions in $\mathcal{H}'_1 \otimes \mathcal{H}'_2 \otimes \mathcal{H}'_3$ that Φ_2 has in $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$. (3.9) and (3.10) follow. ∎

As before, the argument in this proof can be applied to produce a vast range of different finite expansions.

4. Isolation.

Clifton (1994) showed that a dense set of Ψ do not have a triorthogonal decomposition of the form defined by theorem 2.3, and he gave examples of wavefunctions with no decomposition of the form defined in theorem 2.1. He argued that this was enough to make the theorems irrelevant as solutions to the problems of the modal interpretation caused by degeneracies in the Schmidt decomposition. The non-universality of triorthogonal decompositions was also pointed out by Peres (1995). In this section, I shall use continuity arguments to show that there are open sets of wavefunctions containing no elements with various decompositions. I shall also use a thermodynamic argument to suggest that triorthogonal decompositions are not relevant to the measurement problem.

For a state (density matrix) ρ on a Hilbert space $\mathcal{H}$, let $(r_n(\rho))_{n\geq 1}$ be the unique complete decreasing sequence of eigenvalues of ρ, allowing repetitions. Thus, ρ takes the form $\rho = \sum_{n\geq 1} r_n(\rho)|\psi_n\rangle\langle\psi_n|$ for some orthonormal basis $(\psi_n)_{n\geq 1}$ of $\mathcal{H}$ with $1 \geq r_1(\rho) \geq r_2(\rho) \geq \ldots \geq 0$. The entropy $S(\rho)$ is defined by $S(\rho) = -\mathrm{tr}(\rho \log \rho) = -\sum_{n\geq 1} r_n(\rho) \log r_n(\rho)$.

Lemma
4.1 *For each fixed n, $r_n(\rho)$ is continuous in ρ.*
4.2 $0 \leq S(\rho) \leq \log(\dim \mathcal{H})$.
4.3 *On a finite-dimensional space, S is continuous.*

4.4 *On an infinite-dimensional space, S is lower semicontinuous. Thus, at the limit of a convergent sequence, S can jump down but not up.*

proof This is all quite standard. A proof of (4.1) can be found in lemma (2.1) of Bacciagaluppi, Donald, and Vermaas (1995), following a remark of Simon (1973). The lower bound in (4.2) holds because each term in the sum is positive. The upper bound can be proved using the non-positivity (or, according to convention, non-negativity) of relative entropy. This important inequality says that $\mathrm{tr}(-\rho\log\rho + \rho\log\omega) \leq 0$ for all states ω. Choosing for ω the completely mixed state yields the upper bound of (4.2). (4.3) and (4.4) follow from (4.1). On a finite-dimensional space, $S(\rho)$ is a finite sum of continuous functions. On an infinite-dimensional space, $S(\rho)$ is the supremum of the family of continuous non-negative functions corresponding to the partial sums. ∎

If $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$, then let ρ_i (respectively ρ_{ij}) denote the partial trace of ρ on $\mathcal{H}_i$ (resp. on $\mathcal{H}_i \otimes \mathcal{H}_j$).

Lemma 4.5 *Suppose that $\Psi \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$ has a decomposition of the form $\Psi = \sum_{k=1}^K a_k \psi_k^1 \psi_k^2 \psi_k^3$ where K is finite.*
Then $S((|\Psi\rangle\langle\Psi|)_i) \leq \log K$ for $i = 1, 2, 3$.

proof Using the Gram-Schmidt orthogonalization process, it is possible to find an orthonormal basis $(u_n^i)_{n\geq 1}$ for $\mathcal{H}_i$ such that $\{\psi_k^i : k = 1, \ldots, K\}$ is in the span of $\{u_k^i : k = 1, \ldots, K\}$.

Then $(|\Psi\rangle\langle\Psi|)_i$ is a density matrix on the K-dimensional Hilbert space spanned by $\{u_k^i : k = 1, \ldots, K\}$ and so the result is a consequence of (4.2). ∎

Theorem

4.6 *Suppose that $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$ and that $\dim \mathcal{H}_3 > \dim \mathcal{H}_1$. Then there exists $\Phi \in \mathcal{H}$ and $\delta > 0$ such that $||\Phi - \Psi|| > \delta$ for any Ψ with a decomposition of the form $\Psi = \sum_{k=1}^K a_k \psi_k^1 \psi_k^2 \psi_k^3$ where $\{\psi_k^1 : k = 1, \ldots, K\} \subset \mathcal{H}_1$ is a linearly independent set of wavefunctions.*

4.7 *Suppose that $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3 \otimes \mathcal{H}_4$ with $\dim \mathcal{H}_m = N \geq 2$ for $m = 1, 2, 3, 4$. Then there exists $\Phi \in \mathcal{H}$ and $\delta > 0$ such that $||\Phi - \Psi|| > \delta$ for any Ψ with a decomposition of the form $\Psi = \sum_{k=1}^K a_k \psi_k^1 \psi_k^2 \psi_k^3 \psi_k^4$ where at least one of the sets $\{\psi_k^m : k = 1, \ldots, K\}$ is linearly independent.*

proof

For $i = 1, 2, 3$, let $(u_n^i)_{n=1}^{N_i}$ be an orthonormal basis for $\mathcal{H}_i$.

(4.6) In a Hilbert space of dimension N, any set of linear independent vectors can have at most N elements. Thus, by lemma 4.5, if Ψ has a decomposition of the required form, then $S((|\Psi\rangle\langle\Psi|)_3) \leq \log N_1$.

Let

$$\Phi = \left(\sum_{n=1}^{N_1} \frac{1}{\sqrt{N_1 + 1}} u_n^1 u_1^2 u_n^3 \right) + \frac{1}{\sqrt{N_1 + 1}} u_1^1 u_2^2 u_{N_1+1}^3.$$

Then $(|\Phi{>}{<}\Phi|)_3 = \dfrac{1}{N_1+1}\displaystyle\sum_{n=1}^{N_1+1} |u_n^3{>}{<}u_n^3|$ so that $S((|\Phi{>}{<}\Phi|)_3) = \log(N_1+1)$.

As S is lower semicontinuous, there exists $\delta > 0$ such that $||\Phi - \Phi'|| < \delta \Rightarrow S((|\Phi'{>}{<}\Phi'|)_3) \geq \log(N_1 + \frac{1}{2})$.

(4.7) Let $(u_n^4)_{n=1}^N$ be an orthonormal basis for $\mathcal{H}_4$. Applying lemma 4.5 to the triple product $\mathcal{H} = \mathcal{H}_1 \otimes (\mathcal{H}_2 \otimes \mathcal{H}_3) \otimes \mathcal{H}_4$ shows that, if Ψ has a decomposition of the required form, then $S((|\Psi{>}{<}\Psi|)_{23}) \leq \log K$. Linear independence implies that $K \leq N$.

Let

$$\Phi = \left(\sum_{n=1}^{N} \frac{1}{\sqrt{N+1}} u_n^1 u_n^2 u_1^3 u_1^4\right) + \frac{1}{\sqrt{N+1}} u_1^1 u_1^2 u_2^3 u_2^4.$$

Then $(|\Phi{>}{<}\Phi|)_{23} = \dfrac{1}{N+1}\left(\displaystyle\sum_{n=1}^{N} |u_n^2 u_1^3{>}{<}u_n^2 u_1^3| + |u_1^2 u_2^3{>}{<}u_1^2 u_2^3|\right)$ so that $S((|\Phi{>}{<}\Phi|)_{23}) = \log(N+1)$. ∎

This theorem confirms the intuition that a small system provides too few degrees of freedom to allow a spanning set of wavefunctions to be chosen in each situation to correlate with product wavefunctions on a sufficiently large product system. The argument allows many variations. For example, the argument of (4.6) also proves that if $\dim \mathcal{H}_3 > \dim \mathcal{H}_1$ and $\dim \mathcal{H}_3 > \dim \mathcal{H}_2$, then there exists $\Phi \in \mathcal{H}$ and $\delta > 0$ such that $||\Phi - \Psi|| > \delta$ for any Ψ with a decomposition of the form $\Psi = \sum_{k=1}^K a_k \psi_k^1 \psi_k^2 \psi_k^3$ where either $\{\psi_k^1 : k = 1, \ldots, K\} \subset \mathcal{H}_1$ or $\{\psi_k^2 : k = 1, \ldots, K\} \subset \mathcal{H}_2$ is a linearly independent set of wavefunctions. This gets round the variant of theorem 2.1 in which $\{\psi_k^1 : k = 1, \ldots, K\}$ need not be linearly independent. (4.7) extends immediately to products of more than four spaces.

Nevertheless, in spaces with dimensions not satisfying the conditions of the theorem or its variants, dense sets of wavefunctions satisfying the conditions of theorem 2.2 may exist. This holds, for example, for triple products of infinite-dimensional spaces by theorem 3.4. It also holds for triple products of two-dimensional spaces. Acín et al. (2000) show that a dense set in such a space can be represented as a superposition of two triple-product wavefunctions, and, as we have seen above, small variations will make the elements of such products linearly independent.

Lemma 4.8 *Let $\Psi \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$ have a triorthogonal decomposition $\Psi = \sum_{k=1}^K a_k \psi_k^1 \psi_k^2 \psi_k^3$, $|a_k| > 0$ for $k = 1, \ldots, K$, and, for $i = 1, 2, 3$, $\{\psi_k^i : k = 1, \ldots, K\}$ is a orthonormal set of wavefunctions in $\mathcal{H}_i$. Suppose that the terms in the decomposition are ordered so that $|a_1| \geq |a_2| \geq \ldots \geq |a_K|$.*

Then

$$r_k((|\Psi{>}{<}\Psi|)_1) = r_k((|\Psi{>}{<}\Psi|)_2) = r_k((|\Psi{>}{<}\Psi|)_3) = |a_k|^2 \tag{4.9}$$

for $k = 1, \ldots, K$ and

$$S((|\Psi{>}{<}\Psi|)_1) = S((|\Psi{>}{<}\Psi|)_2) = S((|\Psi{>}{<}\Psi|)_3). \tag{4.10}$$

proof This is immediate from the definitions. It is not necessary to assume that K is finite. ∎

(4.9) and (4.10) place strong constraints on triorthogonal wavefunctions. For example, (4.9) and (4.1) show that there is a neighbourhood of the wavefunction Ψ of example 3.1 which contains no triorthogonal wavefunctions. More generally, let T be the set of triorthogonal wavefunctions in an arbitrary Hilbert space. Clifton (1994) pointed out that the complement of T is a dense set, because a small perturbation to the component wavefunctions of a triorthogonal decomposition can change it to a decomposition which is not triorthogonal but which does satisfy theorem 2.2. This perturbed wavefunction is not triorthogonal by the uniqueness of its decomposition.

Let O be the complement of the closure of T. By definition, O is open. We can prove that arbitrary wavefunctions are unlikely to be triorthogonal by proving that O is also dense. There are two ways of doing this. In theorem 5.7, I shall prove that T is closed. It is easier, however, to show that O is dense directly, by using lemma 4.11. This shows that there is a point in O arbitrarily close to every point in T. It follows that arbitrarily close to any wavefunction Φ there is a point in O, either because Φ is itself in O, or because Φ is in T, or because Φ is in $\overline{T}$ in which case Φ is arbitrarily close to wavefunctions in T.

Lemma 4.11 *Let Ψ be a triorthogonal wavefunction and choose $\varepsilon \in (0,1)$. Then there exists $\Psi(\varepsilon)$ with $||\Psi - \Psi(\varepsilon)||^2 \leq 2\varepsilon$ such that no wavefunction in a neighbourhood of $\Psi(\varepsilon)$ is triorthogonal.*

proof There are two cases to be considered. Write $\eta = \sqrt{1-\varepsilon}$ and $\eta' = \sqrt{\varepsilon}$.
I) Suppose $\Psi = \psi_1^1\psi_1^2\psi_1^3$. Then let

$$\Psi(\varepsilon) = \eta\psi_1^1\psi_1^2(\eta\psi_1^3 + \eta'\psi_2^3) + \eta'\psi_2^1\psi_2^2\psi_2^3$$

where ψ_1^i and ψ_2^i are orthogonal wavefunctions for $i = 1,2,3$.

II) Suppose Ψ has a triorthogonal decomposition of the form $\Psi = \sum_{k=1}^K a_k\psi_k^1\psi_k^2\psi_k^3$ where $1 > |a_1| \geq |a_2| \geq \ldots \geq |a_K|$. In this case $|a_2| > 0$.

Set

$$\Psi(\varepsilon) = a_1\psi_1^1\psi_1^2(\eta\psi_1^3 + \eta'\psi_2^3) + a_2\psi_2^1\psi_2^2\psi_2^3 + \sum_{k=3}^K a_k\psi_k^1\psi_k^2\psi_k^3$$

where $\eta = \sqrt{1-\varepsilon}$ and $\eta' = \sqrt{\varepsilon}$.

In both cases, it is straightforward to calculate $||\Psi - \Psi(\varepsilon)||^2$ and $r_1(|\Psi(\varepsilon)><\Psi(\varepsilon)|_i)$ for $i = 1,2,3$, and to show that $||\Psi - \Psi(\varepsilon)||^2 \leq 2\varepsilon$ and that

$$r_1((|\Psi(\varepsilon)><\Psi(\varepsilon)|)_1) = r_1((|\Psi(\varepsilon)><\Psi(\varepsilon)|)_2) \neq r_1((|\Psi(\varepsilon)><\Psi(\varepsilon)|)_3).$$

The result then follows from 4.1 and 4.9. ∎

Using (4.10) and equating von Neumann entropy with thermodynamic entropy would seem to confirm that triorthogonal decompositions do not exist for typical systems consisting of measured objects, measuring devices, and environments. This may, however, appear a slightly curious argument to make in a paper which criticizes as unphysical another mathematical structure on the grounds that it is unstable, because entropy itself is also unstable in infinite dimensional spaces. More precisely, if ρ is any state on an infinite dimensional Hilbert space $\mathcal{H}$, then, in any neighbourhood of ρ there exists a state ρ' with $S(\rho') = \infty$. ρ' can be constructed by mixing with ρ an arbitrarily small amount of any given state with infinite entropy and using the concavity of S. Nevertheless, from a physical point of view, the instability of S is irrelevant because what is important physically is the minimization of local free energy. Along similar lines, it is conceivable that a physically-motivated algorithm could be invoked which would imply that only states with appropriate tridecompositions are physically relevant. I think it implausible that such an idea could work, but the proposals of Spekkens and Sipe (2000) might suggest a starting point.

A triorthogonal decomposition will in general fail to exist because a macroscopic environment will occupy far more degrees of freedom than a microscopic object. Free energy minimization constrains the local quantum state to an effectively finite dimensional space of bounded local energy in which entropy is a physically-revelant finite measure of the number of available degrees of freedom. The approach to local equilibrium is a process of exploring the available state space in a way that rules out the sort of wavefunction correlation expressed by a triorthogonal decomposition unless equation (4.10) happens to hold. But local equilibrium equalizes local temperatures rather than local entropies.

In a paper which is ultimately about collapse or the appearance of collapse, it may seem that an appeal to statistical equilibrium is also rather curious. Perfect statistical equilibrium allows only thermodynamic parameters to be observed. Nevertheless, it seems to me that the states of quantum statistical mechanics almost always provide a better approximation to the correct description of observed macroscopic objects than do the wavefunctions of elementary quantum mechanics, given how little information we usually have about such objects.

Equating von Neumann entropy with thermodynamic entropy does more for us than merely to rule out triorthogonal decompositions. For bipartite systems, a Schmidt decomposition implies equality of von Neumann entropy for the two subsystems. Since a Schmidt decomposition exists for all bipartite pure states, in my view, this implies that the answer to question 1.2 is that it is almost always inappropriate to assume that the state of any macroscopic bipartite system, except possibly the entire universe, is pure. The same conclusion follows if we just equate von Neumann entropy with thermodynamic entropy for the total system. If observed systems have non-zero entropy then they should not be described by pure states.

5. Stability for Triorthogonal Decompositions.

We turn in this section to a detailed analysis of decompositions satisfying theorem 2.3. In theorem 5.6, it will be shown that triorthogonal decompositions are not only unique, but also stable, and in theorem 5.7, it will be shown that the set T of triorthogonal wavefunctions is closed. Given the consequent rarity of triorthogonal wavefunctions, I suspect that the main interest in this section lies in the methods used and in lemma 5.5, rather than in these theorems.

The proofs in this section depend on careful estimations using the following three lemmas. These are essentially standard. Recall that $|| \ ||_1$ denotes the trace norm. Reed and Simon (1972, chapter VI) provides an introduction to trace class operators.

Lemma 5.1 *Let $R = \sum_n r_n|\psi_n><\psi_n|$ and $S = \sum_n s_n|\varphi_n><\varphi_n|$ be eigenvalue decompositions of positive trace class operators with $1 \geq r_1 \geq \ldots \geq r_n \geq \ldots \geq 0$ and $1 \geq s_1 \geq \ldots \geq s_n \geq \ldots \geq 0$. Then $|r_n - s_n| \leq ||R - S||_1$ for $n = 1, 2, \ldots$.*

proof This is lemma 2.1 of Bacciagaluppi, Donald, and Vermaas (1995) without the restriction that $\mathrm{tr}(R) = \mathrm{tr}(S) = 1$ which is not needed for the proof. ∎

Lemma *Let Ψ and Φ be wavefunctions and P be a projection. Then*

$$||P(|\Psi><\Psi| - |\Phi><\Phi|)P||_1 \leq ||\,|\Psi><\Psi| - |\Phi><\Phi|\,||_1 \leq 2||\Psi - \Phi|| \tag{5.2}$$

If, moreover, $<\Psi|\Phi> > 0$, then

$$||\,|\Psi><\Psi| - |\Phi><\Phi|\,||_1 \leq 2||\Psi - \Phi|| \leq \sqrt{2}||\,|\Psi><\Psi| - |\Phi><\Phi|\,||_1 \tag{5.3}$$

proof The first inequality of (5.2) follows from the general result that if A is a trace class operator and B is bounded, then $||AB||_1 \leq ||A||_1||B||$ and $||BA||_1 \leq ||A||_1||B||$.

For wavefunctions Ψ and Φ, $||\Psi - \Phi||^2 = 2 - <\Psi|\Phi> - <\Phi|\Psi> \geq 2(1 - |<\Psi|\Phi>|)$. Explicit diagonalization of $|\Psi><\Psi| - |\Phi><\Phi|$ gives

$$||\,|\Psi><\Psi| - |\Phi><\Phi|\,||_1 = 2\sqrt{1 - |<\Psi|\Phi>|^2}.$$

Thus

$$\begin{aligned}||\,|\Psi><\Psi| - |\Phi><\Phi|\,||_1 &= 2\sqrt{1 - |<\Psi|\Phi>|}\sqrt{1 + |<\Psi|\Phi>|} \\ &\leq 2\sqrt{2(1 - |<\Psi|\Phi>|)} \leq 2||\Psi - \Phi||.\end{aligned}$$

If $<\Psi|\Phi> > 0$, then

$$||\Psi - \Phi||^2 = 2 - <\Psi|\Phi> - <\Phi|\Psi> = 2(1 - |<\Psi|\Phi>|)$$

and so

$$||\,|\Psi><\Psi| - |\Phi><\Phi|\,||_1 = 2\sqrt{1 - |<\Psi|\Phi>|}\sqrt{1 + |<\Psi|\Phi>|} \geq \sqrt{2}||\Psi - \Phi||.$$

∎

Lemma 5.4 *If A is a trace class operator on a tensor product Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, and A_1 is the partial trace of A on $\mathcal{H}_1$, then $||A_1||_1 \leq ||A||_1$.*

proof A proof of this is provided in lemma 4.7 of Bacciagaluppi, Donald, and Vermaas (1995). ∎

At the heart of theorem 2.3 lies the fact that a product wavefunction $\psi^1\psi^2$ cannot be decomposed into a non-trivial sum of products. The next lemma develops that fact by providing explicit bounds. In this lemma, it is not assumed that Ψ or Φ are normalized.

Lemma 5.5 *Let $\Psi = a\psi^1\psi^2$ and $\Phi = \sum_k b_k\varphi_k^1\varphi_k^2$ where ψ^1 and ψ^2 are wavefunctions and, for $i = 1, 2$, $(\varphi_k^i)_k$ are orthonormal sequences of wavefunctions.*

Suppose that $||(|\Psi><\Psi|)_1 - (|\Phi><\Phi|)_1||_1 < \varepsilon'$ and that $1 \geq |a|^2 > 2\varepsilon' > 0$.

Then there exists a unique m such that $|\,|a|^2 - |b_m|^2| < \varepsilon'$ and $|b_{m'}|^2 < \varepsilon'$ for $m \neq m'$.

Suppose further that $||\Psi - \Phi|| < \varepsilon'$ and that $\sqrt{\varepsilon'} \leq |a|\varepsilon/3 < 1$ for some $\varepsilon \in (0, 1)$. Then $||a\psi^1\psi^2 - b_m\varphi_m^1\varphi_m^2|| < \varepsilon$, $|<\psi^1|\varphi_m^1>| > 1 - \varepsilon$, and $|<\psi^2|\varphi_m^2>| > 1 - \varepsilon$.

proof Suppose that $||(|\Psi\rangle\langle\Psi|)_1 - (|\Phi\rangle\langle\Phi|)_1||_1 < \varepsilon'$.

$(|\Psi\rangle\langle\Psi|)_1 = |a|^2|\psi^1\rangle\langle\psi^1|$, $(|\Phi\rangle\langle\Phi|)_1 = \sum_k |b_k|^2|\varphi^1_k\rangle\langle\varphi^1_k|$.

Choose m such that $|b_m| = \max\{|b_k|\}$. By lemma 5.1, $|\,|a|^2 - |b_m|^2| < \varepsilon'$ and $|b_{m'}|^2 < \varepsilon'$ for $m' \neq m$.

If $|a|^2 > 2\varepsilon'$ and $|b_{m'}|^2 < \varepsilon'$, then $|\,|a|^2 - |b_{m'}|^2| = |a|^2 - |b_{m'}|^2 > \varepsilon'$ and so m is unique.

Now suppose also that $||\Psi - \Phi|| < \varepsilon'$ and that $\sqrt{\varepsilon'} \leq |a|\varepsilon/3 < 1$, for some $\varepsilon \in (0, 1)$.

Write $\Phi' = b_m\varphi^1_m\varphi^2_m$ and $\Phi'' = \Phi - \Phi' = \sum_{k\neq m} b_k\varphi^1_k\varphi^2_k$.

By orthogonality, $||\Phi||^2 = ||\Phi'||^2 + ||\Phi''||^2$.

$||\Psi|| = |a|$ and $||\Phi'|| = |b_m|$ so we know that $|\,||\Psi||^2 - ||\Phi'||^2\,| < \varepsilon'$. Thus

$$\begin{aligned}
||\Phi''||^2 &= ||\Phi||^2 - ||\Phi'||^2 = ||\Phi||^2 - ||\Psi||^2 + ||\Psi||^2 - ||\Phi'||^2 \\
&= (||\Phi|| - ||\Psi||)(||\Phi|| + ||\Psi||) + |a|^2 - |b_m|^2 \\
&\leq (||\Psi - \Phi||)(2||\Psi|| + ||\Phi - \Psi||) + \varepsilon' \leq \varepsilon'(2|a| + \varepsilon') + \varepsilon' \leq 4\varepsilon'
\end{aligned}$$

and

$$||a\psi^1\psi^2 - b_m\varphi^1_m\varphi^2_m|| = ||\Psi - \Phi + \Phi''|| \leq ||\Psi - \Phi|| + ||\Phi''|| < \varepsilon' + 2\sqrt{\varepsilon'} < 3\sqrt{\varepsilon'} \leq \varepsilon.$$

$$||\Psi||\,||\Psi - \Phi|| \geq |\langle\Psi|\Psi - \Phi\rangle| = |\,||\Psi||^2 - \langle\Psi|\Phi\rangle| \geq ||\Psi||^2 - |\langle\Psi|\Phi\rangle|$$

so that $|\langle\Psi|\Phi\rangle| \geq ||\Psi||^2 - ||\Psi||\,||\Psi - \Phi|| > |a|(|a| - \varepsilon')$.

But $\langle\Psi|\Phi\rangle = \bar{a}(b_m\langle\psi^1|\varphi^1_m\rangle\langle\psi^2|\varphi^2_m\rangle + \sum_{k\neq m} b_k\langle\psi^1|\varphi^1_k\rangle\langle\psi^2|\varphi^2_k\rangle)$ and so

$$\begin{aligned}
|\langle\Psi|\Phi\rangle| \leq |a|(|b_m||\langle\psi^1|\varphi^1_m\rangle||\langle\psi^2|\varphi^2_m\rangle| + \varepsilon'\sqrt{\textstyle\sum_k |\langle\psi^1|\varphi^1_k\rangle|^2}\sqrt{\textstyle\sum_k |\langle\psi^2|\varphi^2_k\rangle|^2}\ ) \\
\leq |a|(|b_m||\langle\psi^1|\varphi^1_m\rangle||\langle\psi^2|\varphi^2_m\rangle| + \varepsilon')
\end{aligned}$$

Thus, for $|a| > 0$, $|b_m||\langle\psi^1|\varphi^1_m\rangle||\langle\psi^2|\varphi^2_m\rangle| > |a| - 2\varepsilon'$.

$$\begin{aligned}
&|a||\langle\psi^1|\varphi^1_m\rangle||\langle\psi^2|\varphi^2_m\rangle| \\
&\quad = |b_m||\langle\psi^1|\varphi^1_m\rangle||\langle\psi^2|\varphi^2_m\rangle| + (|a| - |b_m|)|\langle\psi^1|\varphi^1_m\rangle||\langle\psi^2|\varphi^2_m\rangle| \\
&\quad > |a| - 2\varepsilon' - |\,|a| - |b_m|\,|.
\end{aligned}$$

Now $\varepsilon' > |\,|a|^2 - |b_m|^2\,|$ implies that $\varepsilon' > |\,|a| - |b_m|\,|(|a| + |b_m|) \geq |\,|a| - |b_m|\,|^2$ and so $\sqrt{\varepsilon'} > |\,|a| - |b_m|\,|$ and

$$|a||\langle\psi^1|\varphi^1_m\rangle||\langle\psi^2|\varphi^2_m\rangle| > |a| - 2\varepsilon' - \sqrt{\varepsilon'} > |a| - 3\sqrt{\varepsilon'}.$$

Using $\sqrt{\varepsilon'} \leq |a|\varepsilon/3 < 1$ yields $|a||\langle\psi^1|\varphi^1_m\rangle||\langle\psi^2|\varphi^2_m\rangle| > |a|(1 - \varepsilon)$ and so $|\langle\psi^i|\varphi^i_m\rangle| > 1 - \varepsilon$ for $i = 1, 2$. ∎

The aim now is to analyse the relationship between the components of two neighbouring wavefunctions Ψ and Φ with triorthogonal decompositions

$$\Psi = \sum_{k=1}^{K} a_k\psi^1_k\psi^2_k\psi^3_k \quad \text{and} \quad \Phi = \sum_{k=1}^{K'} b_k\varphi^1_k\varphi^2_k\varphi^3_k.$$

We shall call a triorthogonal decomposition $\Psi = \sum_{k=1}^{K} a_k\psi^1_k\psi^2_k\psi^3_k$ an "ordered triorthogonal decomposition" if the $|a_k|$ are non-increasing ($|a_1| \geq |a_2| \geq \ldots \geq 0$).

We may also write the decomposition in the form $\Psi = \sum_{m=1}^{M} \hat{a}_m(\sum_{k=1}^{K_m} \psi^1_{mk}\psi^2_{mk}\psi^3_{mk})$, where $(|\hat{a}_m|)_{m=1}^{M}$ is the strictly decreasing sequence of distinct non-zero values for the $|a_k|$ ($|\hat{a}_1| > |\hat{a}_2| > \ldots > 0$). We shall call this a "strictly ordered triorthogonal decomposition".

The complexities of theorem 5.6 and its proof arise because the phases of the component wavefunctions are not determined by the decomposition; because the sums may not be finite; and because the smaller in absolute value the coefficients a_k and b_k, the less the corresponding components need agree for a given difference between the total wavefunctions.

Theorem 5.6 *Let* $\Psi = \sum_{m=1}^{M} \hat{a}_m(\sum_{k=1}^{K_m} \psi^1_{mk}\psi^2_{mk}\psi^3_{mk})$ *be a wavefunction with a strictly ordered triorthogonal decomposition. Choose a finite integer* L *with* $1 \leq L \leq M$ *and* $\varepsilon \in (0, \frac{1}{4})$. *Suppose that* Φ *is a wavefunction with a triorthogonal decomposition of the form* $\Phi = \sum_{k=1}^{K'} b_k\varphi^1_k\varphi^2_k\varphi^3_k$ *and that* $||\Psi - \Phi|| < |\hat{a}_L|^2\varepsilon^2/18$.

Then, for each $m \in \{1, 2, \ldots L\}$ *and* $k \in \{1, \ldots, K_m\}$, *there exists a unique* $k' \in \{1, 2, \ldots K'\}$ *such that* $|\,|\hat{a}_m|^2 - |b_{k'}|^2| < 3\varepsilon$, *such that* $|{<}\psi^i_{mk}|\varphi^i_{k'}{>}| > 1 - \varepsilon$ *for* $i = 1, 2, 3$, *and such that* $||\hat{a}_m\psi^1_{mk}\psi^2_{mk}\psi^3_{mk} - b_{k'}\varphi^1_{k'}\varphi^2_{k'}\varphi^3_{k'}|| < 3\sqrt{\varepsilon}$.

proof Set $\varepsilon' = |\hat{a}_L|^2\varepsilon^2/9$. Then, for $m = 1, 2, \ldots, L$, $|\hat{a}_m|^2 > 2\varepsilon' > 0$ and $\varepsilon' < \sqrt{\varepsilon'} \leq |\hat{a}_m|\varepsilon/3 < 1$.

Let $P_{\psi^3_{mk}}$ be the orthogonal projection from $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$ onto $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \{\psi^3_{mk}\}$.

Then $P_{\psi^3_{mk}}\Psi = \hat{a}_m\psi^1_{mk}\psi^2_{mk}\psi^3_{mk}$ and

$$P_{\psi^3_{mk}}\Phi = \sum_{k'=1}^{K'} b_{k'}{<}\psi^3_{mk}|\varphi^3_{k'}{>}\varphi^1_{k'}\varphi^2_{k'}\psi^3_{mk}.$$

Write $R = |P_{\psi^3_{mk}}\Psi{>}{<}P_{\psi^3_{mk}}\Psi|$ and $S = |P_{\psi^3_{mk}}\Phi{>}{<}P_{\psi^3_{mk}}\Phi|$.

Then, by (5.2) and lemma 5.4, $||\Psi - \Phi|| < \frac{1}{2}\varepsilon' \Rightarrow ||R_1 - S_1||_1 < \varepsilon'$. Also, of course, $||P_{\psi^3_{mk}}\Psi - P_{\psi^3_{mk}}\Phi|| < \frac{1}{2}\varepsilon' < \varepsilon'$ and so, by lemma 5.5, there is a unique k' such that

$$|\,|\hat{a}_m|^2 - |b_{k'}|^2|{<}\psi^3_{mk}|\varphi^3_{k'}{>}|^2| < \varepsilon',$$

such that

$$||\hat{a}_m\psi^1_{mk}\psi^2_{mk} - b_{k'}{<}\psi^3_{mk}|\varphi^3_{k'}{>}\varphi^1_{k'}\varphi^2_{k'}|| < \varepsilon,$$

and such that $|{<}\psi^1_{mk}|\varphi^1_{k'}{>}| > 1 - \varepsilon$ and $|{<}\psi^2_{mk}|\varphi^2_{k'}{>}| > 1 - \varepsilon$.

As $1 - \varepsilon > \frac{1}{\sqrt{2}}$, $\varphi^1_{k'}$ and $\varphi^2_{k'}$ are uniquely determined by these inequalities. Projecting onto $\{\psi^1_{mk}\} \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$ implies that $|{<}\psi^3_{mk}|\varphi^3_{k'}{>}| > 1 - \varepsilon$ and so

$$\begin{aligned}|\,|\hat{a}_m|^2 - |b_{k'}|^2| &\leq |\,|\hat{a}_m|^2 - |b_{k'}|^2|{<}\psi^3_{mk}|\varphi^3_{k'}{>}|^2| + |b_{k'}|^2(1 - |{<}\psi^3_{mk}|\varphi^3_{k'}{>}|^2) \\ &< \varepsilon' + 2(1 - |{<}\psi^3_{mk}|\varphi^3_{k'}{>}|) \leq 3\varepsilon.\end{aligned}$$

For all wavefunctions φ and ψ, $||\varphi - {<}\psi|\varphi{>}\psi||^2 = 1 - |{<}\psi|\varphi{>}|^2$. It follows that

$$||b_{k'}\varphi^1_{k'}\varphi^2_{k'}\varphi^3_{k'} - b_{k'}{<}\psi^3_{mk}|\varphi^3_{k'}{>}\varphi^1_{k'}\varphi^2_{k'}\psi^3_{mk}||^2 = |b_{k'}|^2(1 - |{<}\psi^3_{mk}|\varphi^3_{k'}{>}|^2) \leq 2\varepsilon$$

and so

$$||\hat{a}_m\psi^1_{mk}\psi^2_{mk}\psi^3_{mk} - b_{k'}\varphi^1_{k'}\varphi^2_{k'}\varphi^3_{k'}||$$
$$\leq ||\hat{a}_m\psi^1_{mk}\psi^2_{mk}\psi^3_{mk} - b_{k'}{<}\psi^3_{mk}|\varphi^3_{k'}{>}\varphi^1_{k'}\varphi^2_{k'}\psi^3_{mk}|| + 2\sqrt{\varepsilon} < 3\sqrt{\varepsilon}.$$

∎

Theorem 5.7 Let $(\Psi_n)_{n\geq 1}$ be a sequence of triorthogonal wavefunctions and suppose that $\Psi_n \to \Psi$. Then Ψ is triorthogonal.

proof Choose $\varepsilon \in (0, \frac{1}{4})$.

Suppose that $\Psi_n = \sum_{k=1}^{K_n} a_k(n)\psi^1_k(n)\psi^2_k(n)\psi^3_k(n)$ is a triorthogonal decomposition.

By 5.1, 5.2, and 5.4, for $i = 1, 2, 3$ and all k,

$$r_k((|\Psi_n{>}{<}\Psi_n|)_i) \to r_k((|\Psi{>}{<}\Psi|)_i)$$

as $n \to \infty$.

It follows that

$$r_k((|\Psi{>}{<}\Psi|)_1) = r_k((|\Psi{>}{<}\Psi|)_2) = r_k((|\Psi{>}{<}\Psi|)_3).$$

Set $r_k = r_k((|\Psi{>}{<}\Psi|)_1)$.

Let $(\hat{r}_m)_{m=1}^M$ be the ordered sequence of distinct decreasing values for the non-zero r_k. For any finite integer $L \leq M$, let T_L be the total number of r_k greater than or equal to $\hat{r}_L$.

Choose L such that $\sum_{k=1}^{T_L} r_k \geq 1 - \varepsilon^2$.

If $L = M$, write $\hat{r}_{M+1} = 0$. Let $\alpha = \min\{|\hat{r}_k - \hat{r}_{k+1}| : 1 \leq k \leq L\}$.

$\alpha > 0$. There exists N_1 such that $n \geq N_1$ implies $||\Psi_n - \Psi|| < \alpha/6$.

By 5.1, 5.2, and 5.4, for $n \geq N_1$

$$||(|\Psi{>}{<}\Psi|)_1 - (|\Psi_n{>}{<}\Psi_n|)_1||_1 < \alpha/3$$
$$\text{and} \quad |r_k((|\Psi{>}{<}\Psi|)_1) - r_k((|\Psi_n{>}{<}\Psi_n|)_1)| < \alpha/3 \quad \text{for all } k.$$

This implies that there are exactly T_L values of $r_k((|\Psi_n{>}{<}\Psi_n|)_1)$ which are greater than or equal to $\hat{r}_L - \alpha/3$.

Now set $\varepsilon' = |\hat{r}_L - \alpha/3|^2\varepsilon^2/36\}$.

There exists $N_2 \geq N_1$ such that $n \geq N_2$ implies $||\Psi_n - \Psi|| < \varepsilon'$, and so $m, n \geq N_2$ implies $||\Psi_m - \Psi_n|| < 2\varepsilon'$.

By theorem 5.6, there exists a unique matching of the first T_L terms in the triorthogonal decompositions of Ψ_m and Ψ_n such that, using an ordering compatible with this matching, $|{<}\psi^i_k(m)|\psi^i_k(n){>}| > 1-\varepsilon$ for $i = 1, 2, 3$, $|\,|a_k(m)|^2 - |a_k(n)|^2| < 3\varepsilon$, and

$$||a_k(m)\psi^1_k(m)\psi^2_k(m)\psi^3_k(m) - a_k(n)\psi^1_k(n)\psi^2_k(n)\psi^3_k(n)|| < 3\sqrt{\varepsilon}.$$

It follows that the sequences $(|a_k(n)|)_{n\geq 1}$, $(|\psi^i_k(n){>}{<}\psi^i_k(n)|)_{n\geq 1}$, and $(a_k(n)\psi^1_k(n)\psi^2_k(n)\psi^3_k(n))_{n\geq 1}$ are Cauchy, and hence convergent.

With a suitable choice of labelling, the limit of the $|a_k(n)|$ can be taken to $\sqrt{r_k}$. $|\psi^i_k(n){>}{<}\psi^i_k(n)|$ converges to a pure state. Write $|\psi^i_k(n){>}{<}\psi^i_k(n)| \to |\psi^i_k{>}{<}\psi^i_k|$ where some particular choice of phase is made in the limit wavefunction.

Now choose the phases of the $\psi_k^i(n)$ so that $<\psi_k^i(n)|\psi_k^i> \geq 0$ for all i, k, and n, adjusting the phase of $a_k(n)$ to leave $a_k(n)\psi_k^1(n)\psi_k^2(n)\psi_k^3(n)$ unchanged.

In this case, by (5.3),

$$|\psi_k^i(n)><\psi_k^i(n)| \to |\psi_k^i><\psi_k^i| \Rightarrow \psi_k^i(n) \to \psi_k^i.$$

It follows that $a_k(n)\psi_k^1(n)\psi_k^2(n)\psi_k^3(n)$ converges to $\sqrt{r_k}e^{i\theta_k}\psi_k^1\psi_k^2\psi_k^3$ for some phase θ_k. Setting $a_k = \sqrt{r_k}e^{i\theta_k}$ gives

$$a_k(n)\psi_k^1(n)\psi_k^2(n)\psi_k^3(n) \to a_k\psi_k^1\psi_k^2\psi_k^3$$

and $a_k(n) \to a_k$.

Write $\Phi_L = \sum_{k=1}^{T_L} a_k\psi_k^1\psi_k^2\psi_k^3$ and let $\Phi = \lim_{L\to M}\Phi_L$. Φ is a triorthogonal wavefunction.

Write $\Phi_L(n) = \sum_{k=1}^{T_L} a_k(n)\psi_k^1(n)\psi_k^2(n)\psi_k^3(n)$.

With the value of L chosen earlier, $||\Phi_L - \Phi||^2 = \sum_{k>T_L} r_k \leq \varepsilon^2$.

Using the convergence of a finite number of fixed convergent sequences, there exists $N_3 \geq N_2$, such that $n \geq N_3$ implies that

$$|(||\Phi_L(n) - \Psi_n|| - ||\Phi_L - \Phi||)| = |\sqrt{1 - \sum_{k=1}^{T_L}|a_k(n)|^2} - \sqrt{1 - \sum_{k=1}^{T_L} r_k}| < \varepsilon$$

and that $||\Phi_L(n) - \Phi_L|| < \varepsilon$.

This implies that, for $n \geq N_3$,

$$||\Phi - \Psi|| \leq ||\Phi - \Phi_L|| + ||\Phi_L - \Phi_L(n)|| + ||\Phi_L(n) - \Psi_n|| + ||\Psi_n - \Psi|| < 5\varepsilon.$$

As ε was freely chosen, $\Psi = \Phi$ and the result is proved. ∎

6. Conclusion.

An appropriate goal for a realist interpretation of quantum theory is an algorithm providing an explanation of the appearance of collapse. In the modal interpretation, the proposed algorithm was based on the biorthogonal, or Schmidt, decomposition of a wavefunction on a biproduct space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$. Degeneracies, instabilities, and the analysis of plausible states for thermal systems produce obstacles to this program which, in my opinion, are insurmountable.

Bub (1997) discusses several theorems which could be useful in the development of algorithms for realist interpretations of quantum theory. Theorem 2.1 is among these. In my opinion, the instability demonstrated in section 3 would make worthless any algorithm based purely on the uniqueness provided by theorem 2.1. The results in section 4, in particular (4.10), show that theorem 2.3 is also unlikely to be of fundamental physical significance.

The central purpose of this paper is to argue that the uniqueness provided by the tridecompositional theorem is not a physically relevant way of identifying component wavefunctions in product spaces. Even if it is thought necessary to expand wavefunctions into sums of product wavefunctions, the conditions of theorem 2.1 may simply be too strong. For example, with the definitions of example 3.3, $\Psi(\theta)$ may have a unique expansion for θ small and positive satisfying the conditions of theorem 2.1, but, for almost all purposes, an expansion into the orthonormal product basis

$(\psi^1_{n_1}\psi^2_{n_2}\psi^3_{n_3})_{n_1,n_2,n_3}$ is likely to be more appropriate. This will reveal the closeness of $\Psi(\theta)$ to Ψ and show the weight in $\Psi(\theta)$ of the correlation $\psi^1_1\psi^2_1\psi^3_1$.

More generally, however, I believe that it is a mistake to assume that any physical systems should be described by pure states unless we have good reasons to expect that such descriptions may be valid; as for example with well-controlled and carefully prepared microscopic systems. In particular, I think it is a mistake to try to explain the appearance of collapse in terms of the assumption, at any instant, of pure states for open thermal macroscopic systems like brains or cats or measuring devices.

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